Dipole Equilibrium and Stability

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Abstract. A plasma confined in a dipole field exhibits unique equilibrium and stability properties. In particular, equilibria exist at all β values and these equilibria are found to be stable to ballooning modes when they are interchange stable. When a kinetic treatment is performed at low beta we also find a drift temperature gradient mode which couples to the MHD mode in the vicinity of marginal interchange stability.

1 Introduction

A dipole plasma confinement device may be an attractive fusion power source [1, 2]. To explore the equilibrium and stability of a fusion grade plasma confined in such device the Levitated Dipole Experiment (LDX) [3] is currently being built at M.I.T. In LDX the dipole magnetic field will be created by a floating axisymmetric superconducting current ring. There is no toroidal magnetic field. In such a device the relatively rapid radial variation of the magnetic field strength allows very high plasma pressure near the levitated dipole coil while maintaining a much lower plasma pressure at the outer edge. A critical issue for the dipole fusion concept is the nature of plasma equilibrium and stability at very high core plasma pressure and local plasma beta. In this paper, we discuss models of high beta equilibria that demonstrate that plasmas can be MHD stable in a dipole magnetic field even when the local beta greatly exceeds unity.

Dipole MHD equilibrium and stability have been analyzed both analytically and numerically [4, 5, 6]. Reference [4] investigated the numerically computed free boundary equilibria for a plasma in the field of a floating ring and its stability. In Ref. [5] a useful family of equilibrium solutions to the Grad-Shafranov equation has been found for a point dipole by semi-analytic methods. Furthermore, the MHD stability problem of a dipole equilibrium was reduced to a problem of solving a linear ordinary integro-differential equation Ref. [6]. Both the numerical floating ring equilibria and the point dipole equilibria for isotropic plasma pressure have been found to remain stable to ballooning modes at all beta values when they are interchange stable.

Electrostatic plasma modes for magnetic dipole equilibria (the limit of low plasma pressure) have also been studied kinetically under the high collisionality assumption [7]. These modes are flute-like to leading order in the expansion in $(k_\perp \rho_i)^2 \ll 1$, where ρ_i is the ion Larmor radius and $k_\perp$ is the perpendicular component of the wave vector. The electrostatic mode dispersion relation has two branches - an “MHD-like” branch and a “drift” branch. In the absence of collisional dissipation the stability condition of the “MHD-like” electrostatic mode coincides with the MHD interchange stability condition, while stability of the “drift” mode depends on two independent parameters: $\eta \equiv d\ln T_i/d\ln N_i$ and $d \equiv -d\ln p/d\ln \nu$, where T_i and N_i are the ion temperature and density, p is the total plasma pressure and $\nu = \oint d\ell/B$ with B the magnitude of the magnetic field and ℓ the coordinate along the magnetic field line.

2 Ideal MHD Formulation

2.1 Equilibrium

In a laboratory plasma confined in a levitated dipole device the plasma is expected to be isotropic. Since the currents in the floating ring are toroidal, the magnetic field is entirely poloidal and currents within the plasma are toroidal as well. All magnetic field lines are closed so that "flux" or pressure surfaces are determined by their surfaces of rotation about the symmetry axis. The Grad-Shafranov equation for a dipole is particularly simple in form:

$$\nabla \cdot \left(\frac{\nabla \psi}{R^2} \right) = -\mu_0 \frac{dp}{d\psi}, \tag{1}$$

with ψ the poloidal flux function, $p = p(\psi)$ the plasma pressure, and R the cylindrical radial coordinate. We define the local beta as $\beta = 2\mu_0 p/B^2$.

We have solved the equilibrium equation numerically (for arbitrary beta) [4] to obtain a free boundary solution to Eq. (1) in LDX geometry [8]. For a fixed edge pressure the highest beta value is obtained for a pressure profile that is marginally stable to interchange modes. A high-β equilibrium solution, shown in Fig. 1, is found by choosing a pressure profile that is close to the interchange stability boundary ($p \propto \nu^{-\gamma}$, $\gamma = 5/3$ [9]) and has an edge pressure of $20\,Pa$ which yields $\beta_{edge} = 0.67$. The corresponding peak beta is $\beta_{max} = 25$ at $R = 0.85\,m$ and the plasma has effectively excluded the field in this region. Notice that the equilibrium has shifted outward radially and the plasma is now limited on an outer limiter and not on the magnetic separatrix.

For a point dipole it is possible to obtain relatively simple separable solutions of the Grad-Shafranov equation for both the isotropic [5] and anisotropic [10] pressure cases.

2.2 Stability

The ideal MHD interchange and ballooning stability of the magnetic dipole configuration can be evaluated using the MHD energy principle. Considering interchange modes, for which the relevant perpendicular plasma displacement is constant along a field line, we obtain the general finite β interchange stability condition [5, 6]

$$\frac{1}{p}\frac{dp}{d\psi} + \frac{\gamma}{\nu}\frac{d\nu}{d\psi} < 0. \tag{2}$$

Considering the ballooning stability of a magnetic dipole configuration, we notice that short-wavelength ballooning modes bend magnetic field lines, which, along with plasma compression, has stabilizing influence on the modes. The infinite n integro-differential ballooning equation is [6]

$$B\frac{d}{d\ell}\left(\frac{1}{BR^2}\frac{d\xi}{d\ell}\right) + \mu_0\left(2\kappa_\psi \frac{dp}{d\psi} + \Lambda \frac{\rho}{R^2 B^2}\right)\xi = 4\mu_0 \gamma p \kappa_\psi \frac{\langle \kappa_\psi \xi \rangle_\theta}{1 + \gamma \langle \beta \rangle_\theta}, \tag{3}$$

where ξ is the radial displacement function in the plasma perpendicular displacement $\boldsymbol{\xi}_\perp = (\xi/R^2 B^2)\,\nabla\psi$, $\Lambda \propto \omega^2$ and $\kappa_\psi = \boldsymbol{\kappa}/\nabla\psi$. For a dipole, the field line curvature $\boldsymbol{\kappa} = \boldsymbol{b}\cdot\nabla\boldsymbol{b}$ is in the $\nabla\psi$ direction with $\boldsymbol{b} = \boldsymbol{B}/B$.

As shown in Ref. [9], some key properties of the eigenvalues Λ_j, $j = 0, 1, 2, \ldots$, of Eq. (3) can be determined based on the eigenvalues λ_j of the corresponding homogeneous Sturm-Liouville differential equation; that is, Eq. (3) without the right hand side. The homogenious form of Eq. (3) has a complete set of eigenfunctions ξ_j with corresponding distinct eigenvalues λ_j, that can be arranged as an increasing set for specified boundary conditions. As the dipole system is up-down symmetric, the eigenfunctions of this Sturm-Liouville equation are up-down symmetric or antisymmetric, and we assign even (odd) indices j to the symmetric (antisymmetric) eigenfunctions and eigenvalues in such a way that the smaller index corresponds to the eigenfunction with a smaller eigenvalue. Notice that in general the eigenvalues for the even and odd eigenfunctions form two independent increasing sets as the corresponding boundary conditions are different. It is shown in Ref. [9] that $\lambda_{2j+1} = \Lambda_{2j+1} \leq \lambda_{2j+3} = \Lambda_{2j+3}$ and $\lambda_{2j} \leq \Lambda_{2j} \leq \lambda_{2j+2} \leq \Lambda_{2j+2}$. As a result, $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ leads to $\Lambda_j \geq 0$ and ballooning stability, while $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$ leads to $\Lambda_1 \leq 0$ or $\Lambda_0 \leq 0$ and ballooning instability. For the subtle case $\lambda_0 < 0 < \lambda_2$ and $\lambda_1 = \Lambda_1 > 0$ it is shown in Ref. [9] that the equilibrium is ballooning stable and $\Lambda_0 > 0$ (unstable and $\Lambda_0 < 0$), if it is interchange stable (unstable). Further details of this analysis are given in Ref. [6].

It is shown in Refs. [5, 6] that the point dipole equilibrium of Ref. [5] is always interchange stable. Ballooning stability of this equilibrium was studied in Ref. [6], where it was shown that $\lambda_0 < 0$ and $\lambda_2 > \lambda_1 > 0$, so that the point dipole equilibrium is always ballooning stable because it is interchange stable.

Reference [4] studied interchange and ballooning stability of a high pressure laboratory plasma confined in the field of a circular floating coil. It was found that a numerically obtained high beta MHD equilibrium with a peak local beta of $\beta \sim 10$ and volume averaged beta of $\bar{\beta} \sim 0.5$ having a pressure profile near marginal stability for interchange modes, is ballooning stable for the first antisymmetric ballooning mode of Eq. (3) (for which the right hand side vanishes). Since the lowest symmetric ballooning mode and the interchange mode are identical at marginality [6], LDX would be MHD stable for such equilibria.

When radio frequency heating is used to increase the plasma temperature a mild pressure anisotropy may result. Stronger anisotropies are of interest for space and astrophysical dipole configurations where the dipole field is generated by a dynamo mechanism. Consequently, the interchange and ballooning stability of an anisotropic pressure plasma confined by a dipole magnetic field has also been investigated.

An anisotropic fluid energy principle (which reduces to the isotropic limit) has been derived in Ref. [11] from the Kruskal-Oberman [12] formulation in which the plasma is treated kinetically along the magnetic field and as a fluid across the magnetic field. Anisotropic forms of the interchange stability criterion and of the ballooning mode equation, including plasma compressibility, have been obtained. This stability analysis has been applied to the anisotropic pressure family of point dipole equilibria [10]. The mirror instability or firehose instability set limits on the achievable plasma beta, β_0, when the perpendicular pressure $p_\perp$ is greater (mirror) or less than (firehose) the parallel pressure $p_\parallel$. In Ref. [11] it was found that the point dipole equilibria of Ref. [10] are interchange stable for all plasma betas up to these limits, β_{mm} or β_{fh}, whichever is appropriate. At the same time ballooning modes are stable for all betas up to some critical value, which is below β_{mm} for $1 < p_\perp/p_\parallel < 8$ and is equal to β_{mm} for $p_\perp/p_\parallel > 8$. At modest anisotropy the beta threshold may be observable in the high beta plasmas expected in LDX (for $p_\perp/p_\parallel = 1.2$ the beta limit becomes $\beta_{limit} \approx 6$).

3 Low-β Collisional Interchange Modes

We have shown that MHD predicts that at low beta interchange modes limit the pressure gradients that can be stably maintained. Ideal MHD assumes a particularly simple equation of state and ignores finite Larmor radius effects which can become important for ions. We would expect a more detailed kinetic stability treatment to produce drift waves as well as MHD "fluid" modes. We can shed light on this expectation by considering the stability of electrostatic interchange modes in a collisional plasma [7].

To compare the MHD results with those of kinetic theory we define $\omega_{*pi} \equiv (cT_i n/Z_i e)\times (p^{-1}dp/d\psi)$ and $\omega_{di} \equiv (cT_i n/RBZ_i e)\left[\boldsymbol{e}_\zeta \cdot \boldsymbol{b} \times (\nabla \ln B + \boldsymbol{\kappa})\right]$, so that $\langle \omega_{di} \rangle_\theta = -\,(cT_i n/Z_i e)\times (\nu^{-1}d\nu/d\psi)$, where c is the speed of light, $T_i = T_i(\psi)$ and $Z_i e$ are the ion temperature and charge, and $n \gg 1$ is the toroidal mode number. At low β the MHD stability condition (the interchange stability condition, given by Eq. (2)) can then be written as $\omega_{*pi} < \gamma \langle \omega_{di} \rangle_\theta$ or $d < \frac{5}{3}$, where $d = -d \ln p / d \ln \nu$.

We wish then to solve Boltzmann equation in the high collision frequency limit for both ions and electrons and therefore apply the following orderings:

$$\Omega \gg \omega_b \gg \nu_c \gg \omega_* \sim \omega_d \sim \omega, \tag{4}$$

with Ω the cyclotron frequency, ω_b the bounce frequency, ν_c the collision frequency, ω_* the diamagnetic drift frequency, and ω_d the magnetic curvature drift frequency.

Following the treatment of Ref. [7] we can obtain the following electrostatic plasma dispersion relation:

$$\left(d - \frac{5}{3}\right)\lambda^2 + \frac{5}{9}\left(d\frac{3\eta - 7}{1+\eta} + 5\right) + \left(\frac{\bar{b}}{2}\right)\left[\lambda^4 - \left(d - \frac{5}{3}\right)\lambda^3 - \frac{5}{9}\left(3d\frac{1+2\eta}{1+\eta} + 7\right)\lambda^2 - \frac{5}{9}\left(d\frac{3\eta - 7}{1+\eta} + 5\right)\lambda + \frac{25}{9}\frac{d}{1+\eta}\right] = 0, \tag{5}$$

where $\lambda = \omega / \langle \omega_{di} \rangle_\theta$ and $\bar{b} \equiv \langle k_\perp^2 T_i / M_i \Omega_i^2 \rangle_\theta$, with M_i and Ω_i ion mass and Larmor radius in the field B.

This equation has two classes of solutions - high-frequency or "MHD-like" modes for which $\lambda \gg 1$ and low-frequency or "drift temperature gradient" (or DTG) modes for which $\lambda \sim 1$. The modes are uncoupled for $\bar{b}^{1/2} \ll |d - 5/3|$. The "MHD-like" mode is obtained when the first term of the dispersion relation (5) is balanced by the third term, so that we find

$$\left(\frac{\omega}{\langle \omega_{di} \rangle_\theta}\right)_{MHD} = \pm\left[\frac{2\,(5/3 - d)}{\bar{b}}\right]^{1/2}. \tag{6}$$

This mode is stable (unstable) when $d < 5/3$ ($d > 5/3$), as found earlier from ideal MHD. To consider the DTG modes we neglect the finite Larmor radius terms proportional to $\bar{b}$ in the dispersion relation, and obtain

$$\left(\frac{\omega}{\langle \omega_{di} \rangle_\theta}\right)_{drift} = \pm\sqrt{\frac{5}{9}\frac{5 - d\frac{7-3\eta}{1+\eta}}{\frac{5}{3} - d}}. \tag{7}$$

The dispersion relation (5) in its full form was solved numerically by Kesner [7]. It was found that there is always a stable region for the electrostatic modes near $d = 5/3$ and $\eta = 2/3$. The numerical results are shown in Fig. 2 and are in a good agreement with the preceding analytic predictions except in the vicinity of $d = 5/3$ and $\eta = 2/3$.

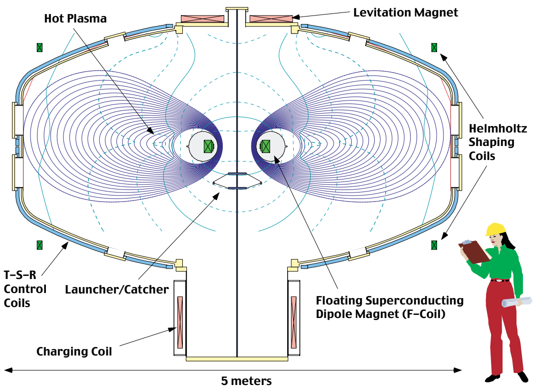

Figure 1: *High* β *equilibrium* (β_{max} = 25) *solution in the LDX geometry.*

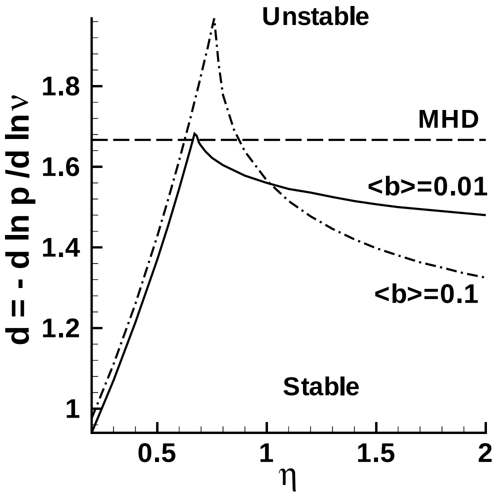

Figure 2: *Exact stability boundaries for* $\bar{b}$*=0.1, 0.01 compared with the ideal MHD boundary.*

Acknowledgements

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