Preface

The guiding principle behind the MFE formulary (the “Formulary” hereafter) is to provide a comprehensive reference for students and researchers working in the field of magnetic confinement fusion. It is a product of the authors’ frustration with searching dozens of textbooks while studying for the MIT doctoral qualifying exams.

The Formulary consists of three broad sections. Chapters 1–2 cover the mathematics, fundamental units, and physical constants relevant to magnetic fusion. Chapters 3–9 cover the basic physics of thermonuclear fusion plasmas, beginning with electrodynamics as a foundation and developing single particle physics, plasma parameters, plasma models, plasma transport, plasma waves, and nuclear physics. Chapters 10–13 cover the physics of toroidally confined plasmas, the fundamentals of magnetic fusion energy, and the parameters for the major magnetic fusion devices of the world.

Much of the content of the Formulary has been derived from an original source, such as peer-reviewed literature, evaluated nuclear data tables, or the pantheon of “standard” mathematics and physics textbooks commonly used in magnetic fusion energy. References are given immediately following the cited item in superscript form as “a:b”, where “a” is the citation number of the reference and “b” is the page number. Full bibliographic entries for all references may be found at the end of the formulary. In addition to providing transparency, this unique feature transforms the Formulary into a gateway to a deeper understanding of the critical equations, derivations, and physics for magnetic fusion energy.

Ultimately, we hope that this work is useful to all those trying to make magnetic fusion energy a reality.

Z & Y
Information

The following sections contain important information regarding the development, use, and distribution of the MFE Formulary. The most up-to-date contact information for the authors of the MFE Formulary is:

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Contributors

The following is a list of people who have contributed their time, effort, and expertise to this project. Their assistance have been critical to continually improving the accuracy and content of the Formulary. Please consider contributing either through our GitHub repository or email!

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Disclaimer

The MFE formulary is by no means complete or error-free. In fact, the statistical laws of the universe essentially guarantee that it has grievous errors and glaring omissions. Thus, we welcome suggestions for additional material, improvements in layout or usability, and corrections to the pesky errors and typos we have tried so hard to eliminate. Even better, we encourage you to obtain the source code from our GitHub repository (see above), make the corrections, and then submit them to us via pull request. You can also reach us by email (see above).

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Acknowledgments

The authors would like to thank Dan Brunner, Chi Gao, Christian Haakonsen, Dr. John Rice, Prof. Anne White, Prof. Dennis Whyte (all of MIT), and Dr. Samuel Cohen (PPPL) for their encouragement, feedback, and proofreading. We would also like thank Heather Barry, Prof. Richard Lester, and Prof. Miklos Porkolab of MIT for their support.

During the writing of the Formulary, the authors were at various times supported by the following funding agencies, to which we would like to extend our gratitude: the MIT Department of Nuclear Science and Engineering, the MIT Plasma Science and Fusion Center, U.S. DoE Grant DE-FG02-94ER54235, U.S. DoE Cooperative Agreement DE-FC02-99ER54512, and the U.S. ORISE Fusion Energy Sciences Program.
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Chapter 1

Mathematics

\( \mathbf{A}, \mathbf{B}, \ldots \), are vector functions

\( \mathbf{T} \) is a tensor

\( \psi \) and \( \xi \) are scalar functions

\( \sigma \) and \( \tau \) refer to surfaces and volumes, respectively

\( d\sigma \) is a differential surface element pointing away from the volume

\( d\tau \) is a differential volume element

\( dr \) is a differential line element

1.1 Vector Identities

1.1.1 Identities Involving Only Vectors

(a) \( \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} \)

(b) \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \)

(c) \( (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \)

(d) \( (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{D}) - \mathbf{D}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}) \)

1.1.2 Identities Involving \( \nabla \)

(a) \( \nabla \cdot (\psi \mathbf{A}) = \psi(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla \psi) \)

(b) \( \nabla \times (\psi \mathbf{A}) = \psi(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla \psi) \)

(c) \( \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \)

(d) \( \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \)

(e) \( \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \)

(f) \( \nabla \cdot (\mathbf{A} \mathbf{B}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B} \)

(g) \( \nabla \cdot (\mathbf{A} \mathbf{T}) = \nabla \psi \cdot \mathbf{T} + \psi \nabla \cdot \mathbf{T} \)

(h) \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \)
(i) \( \nabla^2 A = \nabla (\nabla \cdot A) - \nabla \times \nabla \times A \)

(j) \( \nabla (\psi \xi) = \nabla (\phi \xi) = \psi \nabla \xi + \xi \nabla \psi \)

(k) \( \nabla \cdot (\nabla \psi \times \nabla \xi) = 0 \)

(l) \( \nabla \cdot \nabla \psi = \nabla^2 \psi \)

(m) \( \nabla \times \nabla \psi = 0 \)

1.1.3 Identities Involving \( \int^{12:5} \)

(a) \( \int_{\text{volume}} \nabla \psi \, d\tau = \int_{\text{surface}} \psi \, d\sigma \)

(b) \( \int_{\text{volume}} \nabla \times A \, d\tau = \oint_{\text{surface}} d\sigma \times A \)

(c) \( \int_{\text{surface}} d\sigma \cdot \nabla \times A = \oint_{\text{boundary}} d\mathbf{r} \cdot A \)

(d) \( \oint_{\text{boundary}} d\mathbf{r} \times A = \int_{\text{surface}} (d\sigma \times \nabla) \times A \)

1.2 Curvilinear Coordinate Systems

1.2.1 Cylindrical Coordinates \((r, \theta, z)\) \({12:6-7}^{10}\)

Differential volume: \(d\tau = r \, dr \, d\theta \, dz\)

Relation to cartesian coordinates:

\[
\begin{align*}
    x &= r \cos \theta & \hat{x} &= \cos \phi \hat{r} - \sin \phi \hat{\phi} \\
    y &= r \sin \theta & \hat{y} &= \sin \phi \hat{r} + \cos \phi \hat{\phi} \\
    z &= z & \hat{z} &= \hat{z}
\end{align*}
\]

Unit vector differentials

\[
\frac{d\hat{r}}{dt} = \hat{\theta} \frac{d\theta}{dt}, \quad \frac{d\hat{\theta}}{dt} = -\hat{r} \frac{d\theta}{dt}
\]

Gradient

\[
\nabla \psi = \frac{\partial \psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\theta} + \frac{\partial \psi}{\partial z} \hat{z}
\]
1.2. CURVILINEAR COORDINATE SYSTEMS

Divergence

\[ \nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \]

Curl

\[ \nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{r} \]
\[ + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\theta} \]
\[ + \left( \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) \hat{z} \]

Laplacian

\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \]

Vector-dot-grad

\[ (\mathbf{A} \cdot \nabla) \mathbf{B} = \left( A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_r}{\partial \theta} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r} \right) \hat{r} \]
\[ + \left( A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + A_z \frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r} \right) \hat{\theta} \]
\[ + \left( A_r \frac{\partial B_z}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_z}{\partial \theta} + A_z \frac{\partial B_z}{\partial z} \right) \hat{z} \]

1.2.2 Spherical Coordinates \((r, \theta, \phi)\)

Differential volume: \(dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \)

Relation to cartesian coordinates

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

Gradient

\[ \nabla \psi = \frac{\partial \psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \hat{\phi} \]

Divergence

\[ \nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \]
Curl
\[
\nabla \times \mathbf{A} = \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{1}{r} \frac{\partial A_\theta}{\partial \phi} \right) \mathbf{\hat{r}} + \left( \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r \sin \theta} \frac{\partial}{\partial r} (r A_\phi) \right) \mathbf{\hat{\theta}} + \left( \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) \mathbf{\hat{\phi}}
\]

Laplacian
\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]

Vector-dot-grad
\[
(\mathbf{A} \cdot \nabla) \mathbf{B} = \left( A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_r}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_r}{\partial \phi} + \frac{A_\theta B_\phi - A_\phi B_\theta}{r} \right) \mathbf{\hat{r}} + \left( A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\theta}{\partial \phi} + \frac{A_\theta B_r + A_r B_\theta}{r} \frac{\cos \theta A_\phi B_\phi - \cot \theta A_\phi B_\theta}{r} \right) \mathbf{\hat{\theta}} + \left( A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r \sin \theta} \frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r + A_r B_\phi}{r} \frac{\cot \theta A_\phi B_\theta}{r} \right) \mathbf{\hat{\phi}}
\]

1.3 Integral Relations of Vector Calculus

In this section, let \( \mathbf{A} \equiv \mathbf{A}(x_i, x_j, x_k) \) be a vector function that defines a vector field.

1.3.1 The Fundamental Theorem of Calculus

If \( f(x) \) is a single-valued function on the interval \([a, b]\)
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

1.3.2 Gauss’s (or the Divergence) Theorem

If \( \tau \) is a volume enclosed by a surface \( \sigma \), where \( d\sigma = \mathbf{n} d\sigma \) and \( \mathbf{n} \) is a unit vector pointing away from \( \tau \)
\[
\int_{\text{volume}} (\nabla \cdot \mathbf{A}) \, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\sigma
\]

1.3.3 Stoke’s (or the Curl) Theorem

If \( \sigma \) is an open surface defined by a boundary contour at the surface edge
\[
\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\sigma = \oint_{\text{contour}} \mathbf{A} \cdot dr
\]
1.4 Legendre Polynomials

Legendre’s equation \(14:337\)

\[
(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + l(l + 1)y = 0
\]

\(-1 \leq x \leq 1\) and \(l = 0, 1, 2, \cdots\)

Legendre Polynomials \(14:289\)

<table>
<thead>
<tr>
<th>Order</th>
<th>Corresponding polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l = 0)</td>
<td>(P_0(x) = 1)</td>
</tr>
<tr>
<td>(l = 1)</td>
<td>(P_1(x) = x)</td>
</tr>
<tr>
<td>(l = 2)</td>
<td>(P_2(x) = \frac{1}{2}(3x^2 - 1))</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>(P_3(x) = \frac{1}{2}(5x^3 - 3x))</td>
</tr>
<tr>
<td>(l = 4)</td>
<td>(P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3))</td>
</tr>
<tr>
<td>(l = 5)</td>
<td>(P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x))</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Rodrigues’ formula \(14:286\)

\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l
\]

Orthonormality \(14:286\)

\[
\int_{-1}^{1} P_l(x) P_m(x) \, dx = \int_{0}^{\pi} P_l(\cos \theta) P_m(\cos \theta) \sin \theta \, d\theta = \frac{2}{2l + 1} \delta_{lm}
\]

where \(\delta_{lm}\) is the Kronecker delta: \(l = m, \delta_{lm} = 1; l \neq m, \delta_{lm} = 0.\)

1.5 Bessel Functions

1.5.1 Bessel’s Equation

The most general form of Bessel’s equation is \(14:269\)

\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( \lambda^2 - \frac{p^2}{x^2} \right) y = 0
\]

which has the general solution \(14:270\)

\[
y = AJ_p(\lambda x) + BY_p(\lambda x)
\]

where \(J_p\) are Bessel functions of the first kind and \(Y_p\) are Bessel functions of the second kind (also known as Neumann Functions \(N_p\)), both of order \(p\). Bessel functions of the first kind have no closed form representation;
however, they can be used to define Bessel functions of the second kind:  

\[ Y_p(x) = \frac{J_p(x) \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \]

### 1.5.2 Bessel Function Relations

The following relationships are also valid for \( Y_p(x) \) by replacing \( J_p(x) \) with \( Y_p(x) \):

1. \( J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \)
2. \( \frac{d}{dx}[J_0(x)] = -J_1(x) \)
3. \( \frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \)
4. \( \frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x) \)
5. \( J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \)
6. \( J_{p-1}(x) - J_{p+1}(x) = 2\frac{d}{dx} J_p(x) \)
7. \( \frac{d}{dx} J_p(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x) \)

### 1.5.3 Asymptotic forms of Bessel Functions

For \( x \to \infty \):

\[ J_p(x) \approx \sqrt{\frac{2}{\pi x}} \left[ \cos \left( x - \frac{1}{2}p\pi - \frac{1}{4}\pi \right) \right] \]

\[ Y_p(x) \approx \sqrt{\frac{2}{\pi x}} \left[ \sin \left( x - \frac{1}{2}p\pi - \frac{1}{4}\pi \right) \right] \]

For \( p \to \infty \):

\[ J_p(x) \approx \frac{1}{\sqrt{2\pi p}} \left( \frac{ex}{2p} \right)^p \quad Y_p(x) \approx -\sqrt{\frac{2}{\pi p}} \left( \frac{ex}{2p} \right)^{-p} \]

### 1.6 Modified Bessel Functions

#### 1.6.1 Bessel’s Modified Equation

The most general form of Bessel’s modified equation is:

\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \left( \lambda^2 + \frac{p^2}{x^2} \right) y = 0 \]
1.6. MODIFIED BESSEL FUNCTIONS

which has the general solution \(^{14:275}\)

\[ y = AI_p(\lambda x) + BK_p(\lambda x) \]

where \(I_p\) are Modified Bessel functions of the first kind and \(K_p\) are modified Bessel functions of the second kind. Modified Bessel functions of the first kind have no closed form representation; however, they can be used to define Bessel functions of the second kind: \(^{14:275}\)

\[ K_p(x) = \frac{\pi I_{-p}(x) - I_p(x)}{2 \sin(p\pi)} \]
1.6.2 Modified Bessel Functions Relations

Relations involving $I_p(x)$ \(^{14:280}\)

(a) $xI_{p-1}(x) - xI_{p+1}(x) = 2pI_p(x)$

(b) $I_{p-1}(x) - I_{p+1}(x) = 2 \frac{d}{dx} I_p(x)$

(c) $x \frac{d}{dx} [I_p(x)] + pI_p(x) = xI_{p-1}(x)$

(d) $x \frac{d}{dx} [I_p(x)] - pI_p(x) = xI_{p+1}(x)$

(e) $\frac{d}{dx} [I_0(x)] = I_1(x)$

(f) $I_2(x) = -\frac{2}{x} I_1(x) + I_0(x)$

Relations involving $K_p(x)$ \(^{14:280}\)

(g) $xK_{p-1}(x) - xK_{p+1}(x) = -2pK_p(x)$

(h) $K_{p-1}(x) + K_{p+1}(x) = -2 \frac{d}{dx} K_p(x)$

(i) $x \frac{d}{dx} [K_p(x)] + pK_p(x) = -xK_{p-1}(x)$

(j) $x \frac{d}{dx} [K_p(x)] - pK_p(x) = -xK_{p+1}(x)$

(k) $\frac{d}{dx} [K_0(x)] = -K_1(x)$

(l) $K_2(x) = \frac{2}{x} K_1(x) + K_0(x)$

1.6.3 Asymptotic Forms of Modified Bessel Functions

For $x \to \infty$ \(^{14:278}\)

$$I_p(x) \approx \frac{e^x}{\sqrt{2\pi x}} \left( 1 - \frac{4p^2 - 1}{8x} \right)$$

$$K_p(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left( 1 + \frac{4p^2 - 1}{8x} \right)$$

1.7 Partial Differential Equations

1.7.1 Basis Functions for Laplace’s Equation

Basis functions are the most general solutions to $\nabla^2 \psi = 0$.\(^{18}\)

\(^*\)If m is an integer, $J_{-m} \to Y_m$. If k is imaginary, $J_m(kr) \to I_m(|k|r)$ and $Y_m(kr) \to K_m(|k|r)$

\(^\dagger\) $Y_{lm}(\theta, \phi)$ is the spherical harmonic function
1.7. PARTIAL DIFFERENTIAL EQUATIONS

Plots of $I_p(x)$

Plots of $K_p(x)$

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Basis Function $\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D Cartesian</td>
<td>$\psi = (A \sin kx + B \cos kx) (C e^{ky} + D e^{-ky})$</td>
</tr>
<tr>
<td>2D Cylindrical</td>
<td>$\psi = A_0 + B_0 \ln r + (A \sin n\theta + B \cos n\theta) \times (C r^n + D r^{-n})$</td>
</tr>
<tr>
<td>3D Cartesian</td>
<td>$\psi = A e^{ik_x x} e^{ik_y y} e^{ik_z z}$ with $k_x^2 + k_y^2 - k_z^2 = 0$</td>
</tr>
<tr>
<td>3D Cylindrical</td>
<td>$\psi = A e^{\pm im \theta} e^{\pm k z} J_{\pm m}(kr)$ *</td>
</tr>
<tr>
<td>2D Spherical</td>
<td>$\psi = (Ar^l + Br^{-(l+1)}) P_l(\cos \theta)$</td>
</tr>
<tr>
<td>3D Spherical</td>
<td>$\psi = \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-(l+1)}) Y_{lm}(\theta, \phi)$</td>
</tr>
</tbody>
</table>
1.8 Gaussian Integrals

Definite integral relations of Gaussian integrals:

(a) \[ \int_0^\infty e^{-ax^2} \, dx = \frac{1}{\sqrt{\frac{\pi}{a}}} \]

(b) \[ \int_{-\infty}^\infty e^{-ax^2} \, dx = \left(\frac{\pi}{a}\right)^{1/2} \]

(c) \[ \int_{-\infty}^\infty e^{-ax^2} e^{-2bx} \, dx = \left(\frac{\pi}{a}\right)^{1/2} e^{b^2} \quad \text{for } a>0 \]

(d) \[ \int_{-\infty}^\infty xe^{-a(x-b)^2} \, dx = \left(\frac{\pi}{a}\right)^{1/2} \]

(e) \[ \int_{-\infty}^\infty x^2 e^{-ax^2} \, dx = \frac{1}{\sqrt{\frac{\pi}{a}}} \]

(f) \[ \int_0^\infty x^n e^{-ax^2} \, dx = \begin{cases} \frac{1}{\sqrt{\frac{\pi}{a}}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{(n+1)/2}} & a>0 \\ \frac{(2k-1)!!}{2^{k+1}a^{k+\frac{1}{2}}} \sqrt{\frac{\pi}{a}} & n=2k,a>0 \\ \frac{k!}{2a^{k+\frac{1}{2}}} & n=2k+1,a>0 \end{cases} \]

Definite integrals of common Gaussian relations:

<table>
<thead>
<tr>
<th>n</th>
<th>( \int_0^\infty x^n e^{-ax^2} , dx )</th>
<th>( \int_{-\infty}^\infty x^n e^{-ax^2} , dx )</th>
<th>( \int_0^\infty x^{\frac{n}{2}} e^{-ax} , dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\pi^{1/2}}{2a^{1/2}} )</td>
<td>( \frac{\pi^{1/2}}{a^{1/2}} )</td>
<td>( \frac{\pi^{1/2}}{a^{1/2}} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2a} )</td>
<td>0</td>
<td>( \frac{1}{a} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{\pi^{1/2}}{4a^{3/2}} )</td>
<td>( \frac{\pi^{1/2}}{2a^{3/2}} )</td>
<td>( \frac{\pi^{1/2}}{2a^{3/2}} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{2a^2} )</td>
<td>0</td>
<td>( \frac{1}{a^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{3\pi^{1/2}}{8a^{5/2}} )</td>
<td>( \frac{3\pi^{1/2}}{4a^{5/2}} )</td>
<td>( \frac{3\pi^{1/2}}{4a^{7/2}} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{a^3} )</td>
<td>0</td>
<td>( \frac{2}{a^3} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{15\pi^{1/2}}{16a^{7/2}} )</td>
<td>( \frac{15\pi^{1/2}}{8a^{7/2}} )</td>
<td>( \frac{15\pi^{1/2}}{8a^{7/2}} )</td>
</tr>
</tbody>
</table>

1.9 Error functions

The error function:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \]
Taylor expansion of the error function

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots \right) \]

The complimentary error function

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt = 1 - \text{erf}(x) \]

Taylor expansion of the complimentary error function

\[ \text{erfc}(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}} \left( 1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} - \frac{15}{(2x^2)^3} + \ldots \right) \]
Chapter 2

Fundamental Constants and SI Units

Numerical values for all constants taken from the 2006 CODATA Internationally Recommended Values of the Fundamental Physical Constants.

### Universal Constants

<table>
<thead>
<tr>
<th>Constant</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avagadro’s Constant</td>
<td>$N_A$</td>
<td>6.022 141 79(30)×10$^{23}$</td>
<td>mol$^{-1}$</td>
</tr>
<tr>
<td>Boltzmann’s Constant</td>
<td>$k_B$</td>
<td>1.380 650 4(24)×10$^{-23}$</td>
<td>J/K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.617 343 1(15)×10$^{-5}$</td>
<td>eV/K</td>
</tr>
<tr>
<td>Elementary Charge</td>
<td>$e$</td>
<td>1.602 176 487(40)×10$^{-19}$</td>
<td>C</td>
</tr>
<tr>
<td>Impedance of Vacuum</td>
<td>$Z_0$</td>
<td>376.730 313 461</td>
<td>Ω</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\mu_0/\epsilon_0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Permittivity of Vacuum</td>
<td>$\epsilon_0$</td>
<td>8.854 187 817×10$^{-12}$</td>
<td>F/m</td>
</tr>
<tr>
<td>Permeability of Vacuum</td>
<td>$\mu_0$</td>
<td>4π×10$^{-7}$</td>
<td>N/A$^2$</td>
</tr>
<tr>
<td>Planck’s Constant</td>
<td>$h$</td>
<td>6.626 068 96(33)×10$^{-34}$</td>
<td>J·s</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.135 667 33(10)×10$^{-15}$</td>
<td>eV·s</td>
</tr>
<tr>
<td>H-bar ($h/2\pi$)</td>
<td>$\hbar$</td>
<td>1.054 571 628(53)×10$^{-34}$</td>
<td>J·s</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6.582 118 99(16)×10$^{-16}$</td>
<td>eV·s</td>
</tr>
<tr>
<td>Speed of Light (vacuum)</td>
<td>$c$</td>
<td>2.997 924 58×10$^{8}$</td>
<td>m/s</td>
</tr>
</tbody>
</table>
### Atomic and Nuclear Constants

<table>
<thead>
<tr>
<th>Constant</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron Rest Mass</td>
<td>$m_e$</td>
<td>$9.109 , 382 , 15(45) \times 10^{-31}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5.485 , 799 , 0943(23) \times 10^{-4}$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.510 , 998 , 910(13)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Proton Rest Mass</td>
<td>$m_p$</td>
<td>$1.672 , 621 , 637(83) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.007 , 276 , 466 , 77(10)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$938.272 , 013(23)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Neutron Rest Mass</td>
<td>$m_n$</td>
<td>$1.674 , 927 , 211(84) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.008 , 664 , 915 , 97(43)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$939.565 , 346(23)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Deuteron ($^2$H) Rest Mass</td>
<td>$m_d$</td>
<td>$3.343 , 583 , 20(17) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2.013 , 553 , 212 , 724(78)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1875.612 , 793(47)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Triton ($^3$H) Rest Mass</td>
<td>$m_t$</td>
<td>$5.007 , 355 , 88(25) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3.015 , 500 , 7134(25)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2808.9209 , 06(70)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Helion ($^3$He) Rest Mass</td>
<td>$m_h$</td>
<td>$5.006 , 411 , 92(25) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3.014 , 932 , 2473(26)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2808.391 , 383(70)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Alpha ($^4$He) Rest Mass</td>
<td>$m_\alpha$</td>
<td>$6.644 , 656 , 20(33) \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$4.001 , 506 , 179 , 127(62)$</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3727.379 , 109(93)$</td>
<td>MeV/c$^2$</td>
</tr>
<tr>
<td>Proton to Electron Mass Ratio</td>
<td>$m_p/m_e$</td>
<td>$1836.152 , 672 , 47(80) \approx 6\pi^5$</td>
<td></td>
</tr>
<tr>
<td>Bohr Radius</td>
<td>$a_0$</td>
<td>$0.529 , 177 , 208 , 59(36) \times 10^{-10}$</td>
<td>m</td>
</tr>
<tr>
<td>Classical Electron Radius</td>
<td>$r_e$</td>
<td>$2.817 , 940 , 289 , 4(58) \times 10^{-15}$</td>
<td>m</td>
</tr>
<tr>
<td>Inverse Fine Structure Constant</td>
<td>$1/\alpha$</td>
<td>$137.035 , 999 , 679(94)$</td>
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</tr>
<tr>
<td>Rydberg Constant</td>
<td>$R_\infty$</td>
<td>$1.097 , 373 , 156 , 852 , 7(73) \times 10^7$</td>
<td>m$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$R_\infty , \hbar , c$</td>
<td>$13.605 , 691 , 93(34)$</td>
<td>eV</td>
</tr>
<tr>
<td>Stefan-Boltzmann Constant</td>
<td>$\sigma$</td>
<td>$5.670 , 400 , 40(40) \times 10^{-8}$</td>
<td>W/m$^2$/K$^4$</td>
</tr>
<tr>
<td>Thomson Cross Section</td>
<td>$\sigma_{\text{Th}}$</td>
<td>$0.665 , 245 , 855 , 8(27) \times 10^{-28}$</td>
<td>m$^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.665 , 245 , 855 , 8(27)$</td>
<td>barns</td>
</tr>
</tbody>
</table>
## The System of International Units

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>SI Unit</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity</td>
<td>𝐴</td>
<td>Becquerel</td>
<td>s⁻¹</td>
</tr>
<tr>
<td>Capacitance</td>
<td>C</td>
<td>farad (F)</td>
<td>s²·C²/kg·m²</td>
</tr>
<tr>
<td>Charge</td>
<td>q</td>
<td>*coulomb (C)</td>
<td>C</td>
</tr>
<tr>
<td>Conductance</td>
<td>σ</td>
<td>siemens (S)</td>
<td>s²·C²/kg·m²</td>
</tr>
<tr>
<td>Conductivity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Current</td>
<td>I</td>
<td>ampère (A)</td>
<td>s/C</td>
</tr>
<tr>
<td>Displacement</td>
<td>D</td>
<td>coulomb/m²</td>
<td>m</td>
</tr>
<tr>
<td>Electric Field</td>
<td>E</td>
<td>volt/meter</td>
<td>kg·m²/s²·C</td>
</tr>
<tr>
<td>Electromotance</td>
<td>ε</td>
<td>volt (V)</td>
<td>kg·m²/s²·C</td>
</tr>
<tr>
<td>Energy</td>
<td>W</td>
<td>joule (J)</td>
<td>kg·m²/s</td>
</tr>
<tr>
<td>Force</td>
<td>F</td>
<td>newton (N)</td>
<td>kg/m²/s²</td>
</tr>
<tr>
<td>Frequency</td>
<td>𝜔</td>
<td>hertz (Hz)</td>
<td>s⁻¹</td>
</tr>
<tr>
<td>Impedance</td>
<td>Z</td>
<td>ohm (Ω)</td>
<td>kg·m²/s·C</td>
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<tr>
<td>Inductance</td>
<td>L</td>
<td>henry (H)</td>
<td>kg·m²/s²</td>
</tr>
<tr>
<td>Length</td>
<td>l</td>
<td>*meter (m)</td>
<td>m</td>
</tr>
<tr>
<td>Magnetic Flux</td>
<td>Φ</td>
<td>weber (Wb)</td>
<td>kg·m²/s·C</td>
</tr>
<tr>
<td>Magnetic Flux Density</td>
<td>B</td>
<td>tesla (T)</td>
<td>kg·m²/s·C</td>
</tr>
<tr>
<td>Magnetic Moment</td>
<td>μ</td>
<td>ampere-m²</td>
<td>m²·C²/s</td>
</tr>
<tr>
<td>Magnetization</td>
<td>M</td>
<td>ampere-turn/m</td>
<td>s/m</td>
</tr>
<tr>
<td>Permeability</td>
<td>μ</td>
<td>henry/meter</td>
<td>kg·m²</td>
</tr>
<tr>
<td>Permittivity</td>
<td>ε</td>
<td>farad/meter</td>
<td>kg·m²/s²·C</td>
</tr>
<tr>
<td>Polarization</td>
<td>P</td>
<td>coulomb/m²</td>
<td>m</td>
</tr>
<tr>
<td>Electric Potential</td>
<td>V</td>
<td>volt (V)</td>
<td>kg·m²/s²·C</td>
</tr>
<tr>
<td>Power</td>
<td>P</td>
<td>watt (W)</td>
<td>kg·m²/s³</td>
</tr>
<tr>
<td>Pressure</td>
<td>p</td>
<td>pascal (Pa)</td>
<td>kg/m.s</td>
</tr>
<tr>
<td>Resistance</td>
<td>R</td>
<td>ohm (Ω)</td>
<td>kg·m²/s²·C</td>
</tr>
<tr>
<td>Resistivity</td>
<td>η</td>
<td>ohm-meter</td>
<td>kg·m³/s·C²</td>
</tr>
<tr>
<td>Temperature</td>
<td>T</td>
<td>*kelvin (K)</td>
<td>K</td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>κ</td>
<td>watt/meter/kelvin</td>
<td>kg·m³/s³</td>
</tr>
<tr>
<td>Time</td>
<td>s</td>
<td>*second</td>
<td>s</td>
</tr>
<tr>
<td>Velocity</td>
<td>v</td>
<td>meter/second</td>
<td>m/s</td>
</tr>
</tbody>
</table>

* denotes a fundamental SI base unit
### Energy Conversion Factors

<table>
<thead>
<tr>
<th>Conversion</th>
<th>Conversion Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy $\leftrightarrow$ Temperature</td>
<td>$1 \text{ eV} = 1.602189 \times 10^{-19} \text{ J}$</td>
</tr>
<tr>
<td></td>
<td>$1 \text{ eV} = 1.1604505 \times 10^4 \text{ K}$</td>
</tr>
<tr>
<td>Energy $\leftrightarrow$ Mass</td>
<td>$1 \text{ u} = 931.501 \text{ MeV}/c^2 = 1.660566 \times 10^{-27} \text{ kg}$</td>
</tr>
<tr>
<td>Energy $\leftrightarrow$ Wavelength</td>
<td>$\hbar c = 1239.8419 \text{ MeV}-\text{fm} = 1239.8419 \text{ eV}-\text{nm}$</td>
</tr>
<tr>
<td></td>
<td>$\hbar c = 197.329 \text{ MeV}-\text{fm} = 197.329 \text{ eV}-\text{nm}$</td>
</tr>
<tr>
<td></td>
<td>$e^2/4\pi\epsilon_0 = 1.439976 \text{ MeV}-\text{fm}$</td>
</tr>
</tbody>
</table>
Chapter 3

Electricity and Magnetism

In this chapter, all units are SI.

\( e \) is the elementary electric charge  
\( q \) is the total particle charge  
\( Z \) is the particle atomic (proton) number  
\( n \) is the particle density  
\( U \) is energy  
\( r \) is the particle position  
\( v \) is the particle velocity  
\( \rho \) is volumetric charge density  
\( \sigma \) is surface charge density  
\( J \) is volumetric current density  
\( K \) is surface current density  
\( \tau \) and \( \sigma \) are the volume and surface, respectively  
\( d\tau \), \( d\sigma \), and \( dl \) are the volume, surface, and line elements, respectively  
\( b \) and \( f \) subscripts refer to bound and free charges

3.1 Electromagnetic in Vacuum

3.1.1 Fundamental Equations

Maxwell’s equations \(^{10,326}\):

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\]

Electrostatic scalar potential relations \(^{10,87}\):

\[
\mathbf{E} = -\nabla V \\
\nabla^2 V = -\frac{\rho}{\epsilon_0} \\
V = -\int \mathbf{E} \cdot dl \\
V = \frac{1}{4\pi \epsilon_0} \int \frac{\rho}{r} d\tau
\]
Electrostatic vector potential relations\textsuperscript{10,240}

\[ B = \nabla \times A \]
\[ \nabla^2 A = -\mu_0 J \]
\[ A = \frac{\mu_0}{4\pi} \int \frac{J}{r} d\tau \]

Electromagnetic energy stored in the fields\textsuperscript{10,348}

\[ U_{\text{em}} = \frac{1}{2} \int \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \]

Coulomb Force\textsuperscript{10,59}

\[ F = \frac{q_1 q_2}{4\pi \epsilon_0 |r_1 - r_2|^2} \frac{r_1 - r_2}{|r_1 - r_2|} \]

Lorentz force law\textsuperscript{10,204}

\[ F = q \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \]

Biot-Savart law\textsuperscript{10,215}

\[ \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{r}}{r^2} \, dl \]

3.1.2 Boundary Conditions

For given surface \( S \), + and - refer to above and below \( S \), respectively. \( \hat{n} \) is a unit vector perpendicular to \( S \).

Electrostatic boundary conditions on \( \mathbf{E} \)\textsuperscript{10,179}

\[ E_{\|}^+ - E_{\|}^- = \sigma / \epsilon_0 \]
\[ E_{\perp}^+ - E_{\perp}^- = 0 \]

Magnetostatic boundary conditions on \( \mathbf{B} \)\textsuperscript{10,241}

\[ B_{\perp}^+ - B_{\perp}^- = 0 \]
\[ B_{\|}^+ - B_{\|}^- = \mu_0 K \]
\[ \mathbf{B}_+ - \mathbf{B}_- = \mu_0 (\mathbf{K} \times \hat{n}) \]
3.2 Electromagnetics in Matter

3.2.1 Fundamental Equations

Maxwell’s equations in matter

\[ \nabla \cdot \mathbf{D} = \rho_f \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \]

The polarization in linear media (\( \chi_e \) is the polarizability)

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \]

[electric dipole moments per m\(^{-3}\)]

The magnetization in linear media (\( \chi_m \) is the magnetization)

\[ \mathbf{M} = \chi_m \mathbf{H} \]

[electric dipole moments per m\(^{-3}\)]

The displacement field

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]
\[ = \varepsilon \mathbf{E} \quad \text{linear media only where} \quad \varepsilon \equiv \varepsilon_0 (1 + \chi_e) \]

The H-field (Magnetic field)

\[ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \]
\[ = \frac{1}{\mu} \mathbf{B} \quad \text{linear media only where} \quad \mu \equiv \mu_0 (1 + \chi_m) \]

Associated bound charges (\( \sigma_b, \rho_b \)) and currents (\( \mathbf{K}_b, \mathbf{J}_b \))

\[ \sigma_b = \mathbf{P} \cdot \mathbf{n} \]
\[ \rho_b = -\nabla \cdot \mathbf{P} \]
\[ \mathbf{K}_b = \mathbf{M} \times \mathbf{n} \]
\[ \mathbf{J}_b = \nabla \times \mathbf{M} \]

3.2.2 Boundary Conditions

For given surface \( \mathcal{S} \), + and - refer to above and below \( \mathcal{S} \), respectively. \( \mathbf{n} \) is a unit vector perpendicular to \( \mathcal{S} \).

\[ D_+^\perp - D_-^\perp = \sigma_f \]
\[ D_+^\parallel - D_-^\parallel = \mathbf{P}_+^\parallel - \mathbf{P}_-^\parallel \]
\[ H_+^\perp - H_-^\perp = -\left( M_+^\perp - M_-^\perp \right) \]
\[ H_+^\parallel - H_-^\parallel = \mathbf{K}_f \times \mathbf{n} \]

3.3 Dipoles

In this section, \( \mathbf{p} \) and \( \mathbf{m} \) are electric and magnetic dipoles, respectively. \( \mathbf{N} \) is the torque and \( \mathbf{F} \) is the force generated by the dipole.
### Chapter 3. Electricity and Magnetism

#### Definition

\[ p \equiv \int r \rho(r) d\tau \]

\[ E_{dip}(r) = \frac{1}{4\pi\varepsilon_0 r^3} [3(p \cdot \hat{r})\hat{r} - p] \]

\[ E_{dip}(r, \theta) = \frac{p}{4\pi\varepsilon_0 r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \]

\[ m \equiv I \int d\sigma \]

\[ B_{dip} = \frac{\mu_0}{4\pi r^3} [3(m \cdot \hat{r})\hat{r} - m] \]

\[ B_{dip}(r, \theta) = \frac{\mu_0 m}{4\pi r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \]

#### Fields

\[ F = (p \cdot \nabla) E \]

\[ N = p \times E \]

\[ U = -p \cdot E \]

#### Potentials

\[ V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{p \cdot \hat{r}}{r^2} \]

\[ A_{dip}(r) = \frac{\mu_0}{4\pi} \frac{m \times \hat{r}}{r^2} \]

### 3.4 Circuit Electrodynamics

Microscopic Ohm’s law

\[ \mathbf{J} = \sigma_c \mathbf{E} \]

where \( \sigma_c \) is the conductivity. Resistivity, \( \rho_r \), is defined as \( \rho_r = 1/\sigma_c \).

Macroscopic Ohm’s law

\[ V = IR \]

where \( V \) is the voltage, \( I \) is the current, and \( R \) is the resistance.

The voltage due to a changing magnetic field (Faraday’s Law),

\[ V = -\frac{d\Phi}{dt} \quad \Phi = \int_{\text{surface}} \mathbf{B} \cdot d\mathbf{A} \]

Capacitance is written as \( C \), and inductance is written as \( L \).

\[ Q = CV \quad \Phi = LI \]

\[ I = -C \frac{dV}{dt} \quad V = -L \frac{dI}{dt} \]

Energy stored in capacitance and inductance

\[ U = \frac{1}{2} LI^2 + \frac{1}{2} CV^2 \]
3.5 Conservation Laws

Conservation of charge \( \frac{\partial \rho}{\partial t} = -\nabla \cdot J \)

Poynting vector \( S \equiv \frac{1}{\mu_0} (E \times B) \)

Poynting’s theorem (integral form) \( \frac{dU}{dt} = -\frac{d}{dt} \int_{\text{volume}} \left( \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \right) d\tau - \frac{1}{\mu_0} \oint_{\text{surface}} (E \times B) \cdot d\sigma \)

Poynting’s theorem (differential form) \( \frac{\partial}{\partial t} (U_{\text{mechanical}} + U_{\text{em}}) = -\nabla \cdot S \)

Maxwell’s stress tensor \( T_{ij} \equiv \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \)

Electromagnetic force density on collection of charges \( f = \nabla \cdot \vec{T} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \vec{S} \)

Total electromagnetic force on collection of charges \( \vec{F} = \oint_{\text{surface}} \vec{T} \cdot d\sigma - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\text{volume}} \vec{S} d\tau \)

Momentum density in electromagnetic fields \( p_{\text{em}} = \mu_0 \epsilon_0 \vec{S} \)

Conservation of momentum in electromagnetic fields \( \frac{\partial}{\partial t} (p_{\text{mech}} + p_{\text{em}}) = \nabla \cdot \vec{T} \)

3.6 Electromagnetic Waves

In this section,

\( \lambda \) is the wavelength
\( k = 2\pi / \lambda \) is the wave number
\( \nu \) is the frequency
\( \omega = 2\pi \nu \) is the angular frequency
\( T = 1/\nu \) is the period
\( \mathbf{k} = \hat{\mathbf{k}} \mathbf{k} \) is the wave number vector
\( \hat{\mathbf{n}} \) is the polarization vector in the direction of electric field
\( \mathbf{X} \) is a complex vector

The wave equation in three dimensions
\[
\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}
\]
is satisfied by two transformations of Maxwell’s equations in vacuum
\[
\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\]
These have sinusoidal solutions, but it is more convenient to work with imaginary exponentials and take the real parts
\[
\mathbf{E}(\mathbf{r}, t) = E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \hat{\mathbf{n}}
\]
\[
\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} (\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E}
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Averaged energy per unit volume</td>
<td>\langle u \rangle</td>
<td>\frac{1}{2} \epsilon_0 E_0^2</td>
</tr>
<tr>
<td>Averaged energy flux density</td>
<td>\langle S \rangle</td>
<td>\frac{1}{2} \epsilon_0 E_0^2 \mathbf{k}</td>
</tr>
<tr>
<td>Averaged momentum density</td>
<td>\langle P \rangle</td>
<td>\frac{1}{2} \epsilon_0 E_0^2 \mathbf{k}</td>
</tr>
<tr>
<td>Intensity</td>
<td>I</td>
<td>\langle S \rangle</td>
</tr>
<tr>
<td>Radiation pressure</td>
<td>P</td>
<td>\frac{I}{c}</td>
</tr>
</tbody>
</table>

### 3.6.1 EM Waves in Matter

In this section, \( \theta \) is measured from the normal to the surface

Assuming no free charge or current in a linear media, the EM wave equations become
\[
\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]
\[
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{E}}{\partial t}
\]
3.6. ELECTROMAGNETIC WAVES

Speed of light in a material \( \text{10:383} \)

\[
v = \frac{1}{\sqrt{\varepsilon \mu}} = \frac{c}{n}
\]

Index of refraction \( \text{10:383} \)

\[
n \equiv \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \approx \sqrt{\varepsilon_r}
\]

Intensity \( \text{10:383} \)

\[
I = \frac{1}{2} \varepsilon_0 v E_0^2
\]

Boundary conditions at a material surface \( \text{10:384} \)

\[
\varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp \quad \quad \quad E_1^\parallel = E_2^\parallel \\
B_1^\perp = B_2^\perp \quad \quad \quad \quad \frac{1}{\mu_1} B_1^\parallel = \frac{1}{\mu_2} B_2^\parallel
\]

Reflection and transmission coefficients \( \text{10:386} \)

\[
R \equiv \frac{I_{\text{ref}}}{I_{\text{inc}}} = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \\
T \equiv \frac{I_{\text{trans}}}{I_{\text{inc}}} = \frac{4n_1 n_2}{(n_1 + n_2)^2} \\
R + T = 1
\]

Snell’s Laws for oblique incidence on material surface \( \text{10:388} \)

\[
k_{\text{inc}} \sin \theta_{\text{inc}} = k_{\text{ref}} \sin \theta_{\text{ref}} = k_{\text{trans}} \sin \theta_{\text{trans}}
\]

\[
\theta_{\text{inc}} = \theta_{\text{ref}}
\]

\[
\sin \theta_{\text{trans}} = \frac{n_1}{n_2}
\]

Fresnel Equations \( \text{13:305–306} \)

<table>
<thead>
<tr>
<th>Polarization to incident plane</th>
<th>( E_{\text{trans}}/E_{\text{inc}} )</th>
<th>( E_{\text{ref}}/E_{\text{inc}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpendicular</td>
<td>( \frac{2n_1 \cos \theta_{\text{inc}}}{n_1 \cos \theta_{\text{inc}} + (\mu_1/\mu_2) \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}} )</td>
<td>( \frac{n_1 \cos \theta_{\text{inc}} - (\mu_1/\mu_2) \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}}{n_1 \cos \theta_{\text{inc}} + (\mu_1/\mu_2) \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}} )</td>
</tr>
<tr>
<td>Parallel</td>
<td>( \frac{2n_1 n_2 \cos \theta_{\text{inc}}}{(\mu_1/\mu_2) n_2 \cos \theta_{\text{inc}} + n_1 \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}} )</td>
<td>( \frac{(\mu_1/\mu_2) n_2 \cos \theta_{\text{inc}} - n_1 \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}}{(\mu_1/\mu_2) n_2 \cos \theta_{\text{inc}} + n_1 \sqrt{n_2^2 - n_1^2 \sin^2 \theta_{\text{inc}}}} )</td>
</tr>
</tbody>
</table>

Brewster’s angle (no reflection of perpendicular incidence wave) \( \text{10:390} \)

\[
\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}
\]
where \( \beta = \mu_1 n_2 / \mu_2 n_1 \)

If the wave is in a conductor, it will experience damping due to the presence of free charges, subject to \( \mathbf{J}_f = \sigma \mathbf{E} \). Solving Maxwell’s equations gives

\[
\tilde{k}^2 = \mu \epsilon \omega^2 + i \mu \sigma \omega
\]

Decomposing gives real and imaginary parts of the wave vector \( \tilde{k} = k + i \kappa \)

\[
k \equiv \omega \sqrt{\frac{\epsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} + 1 \right]^{1/2}} \quad \kappa \equiv \omega \sqrt{\frac{\epsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right]^{1/2}}
\]

Knowing the imaginary part of the wave number, allows to know the damping of the wave which is characterized by a skin depth or the e-folding length \( d \equiv 1 / \kappa \).

### 3.7 Electrodynamics

We are allowed to choose \( \nabla \cdot \mathbf{A} \); two common gauges used are the Lorentz and Coulomb gauge. \( \text{10:421–422} \)

<table>
<thead>
<tr>
<th>Gauge</th>
<th>( \nabla \cdot \mathbf{A} )</th>
<th>V Equation</th>
<th>A Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorentz</td>
<td>( -\mu_0 \epsilon_0 \frac{\partial V'}{\partial t} )</td>
<td>( \nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V'}{\partial t^2} = -\frac{\rho}{\epsilon_0} )</td>
<td>( \nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} )</td>
</tr>
<tr>
<td>Coulomb</td>
<td>0</td>
<td>( \nabla^2 V = -\frac{\rho}{\epsilon_0} )</td>
<td>( \nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \nabla \left( \frac{\partial V}{\partial t} \right) )</td>
</tr>
</tbody>
</table>

For any scalar function \( \lambda \), any potential formulation is valid if \( \text{10:420} \)

\[
\mathbf{A}' = \mathbf{A} + \nabla \lambda \\
V' = V - \frac{\partial \lambda}{\partial t}
\]

### 3.7.1 Fields of Moving Charges

In this section, \( \|A\| \) evaluates \( A \) at the retarded time. \( \text{10:423} \)

Definition of retarded time \( \text{10:423} \)

\[
t_{\text{ret}} \equiv t - \frac{|\mathbf{r}(t) - \mathbf{r}(t_{\text{ret}})|}{c}
\]

The Lienard-Wiechert potentials \( \text{10:432–433} \)

\[
V = \frac{q}{4 \pi \epsilon_0} \left\| \frac{1}{r \kappa} \right\| \\
\mathbf{A} = \frac{\mu_0 q}{4 \pi} \left\| \frac{\mathbf{V}}{r \kappa} \right\|
\]
Electric field of a moving point charge

\[ E = \frac{q}{4\pi\varepsilon_0} \left| \left( \hat{r} - \frac{\mathbf{v}}{c} \right) \left( 1 - \frac{v^2}{c^2} \right) - \hat{r} \times (\hat{r} - \frac{\mathbf{v}}{c}) \times \frac{\mathbf{a}}{c} \right| \]

Magnetic field of a moving point charge

\[ \mathbf{B} = \frac{|\hat{r}|}{c} \times \mathbf{E} \]

where \( \kappa = 1 - \hat{r} \cdot \frac{\mathbf{v}}{c} \).

### 3.7.2 Radiation by Charges

In this section, \( \mathbf{u} \equiv c\hat{r} - \mathbf{v} \), \( \gamma \equiv 1/\sqrt{1 - v^2/c^2} \) is the relativistic gamma factor, and \( \mathbf{a} \) is the acceleration.

Poynting vector associated with a moving charge

\[ \mathbf{S} = \frac{1}{\mu_0 c} \left[ E^2 \hat{r} - (\hat{r} \cdot \mathbf{E}) \mathbf{E} \right] \]

Non-relativistic power radiated by a moving charge

\[ P = \frac{\mu_0 q^2 a^2}{6\pi c} \]

Relativistic power radiated by a moving charge per solid angle

\[ \frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0} \frac{|\hat{r} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{r} \cdot \mathbf{u})^6} \]

Relativistic power radiated by a moving charge

\[ P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left( a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right) \]

Relativistic force of a moving charge

\[ \mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c} \frac{d\mathbf{a}}{dt} \]
Chapter 4

Single Particle Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

- $e$ is the elementary electric charge
- $q$ is the total particle charge
- $Z$ is the particle atomic (proton) number
- $m$ is the particle mass
- $r$ is the particle position
- $v$ is the particle velocity
- $U$ is energy
- $T$ is temperature; $T_{keV} = T$ in units of kiloelectron-volts
- $E$ and $B$ are the electric and magnetic fields
- $\hat{b}$ is a unit vector in the direction of $B$
- $||$ and $\perp$ indicate parallel and perpendicular to $\hat{b}$

4.1 Single Particle Motion in $E$ and $B$ Fields

4.1.1 General Formulation

Single particle trajectories result from solving Newton’s second law for a particle with charge $q$ and mass $m$ in electric and magnetic fields: \[ m\frac{dv}{dt} = q(E + v \times B) \quad \frac{dr}{dt} = v \]

If $E$ and $B$ are independent of time, the particle’s kinetic and potential energy is conserved \[ \frac{1}{2}mv^2 + qV = \text{constant} \]

where $V$ is the scalar potential ($E = -\nabla V$).

4.1.2 Gyro Motion Solutions for $B = B_0\hat{z}$; $E = 0$

Particle initially has $r = (x_0, y_0, z_0)$, $v = v_\parallel + v_\perp$, and arbitrary phase, $\phi$. 

35
Parallel to the field: \( z(t) = z_0 + v_\parallel t \)

Perpendicular to the field: \( x(t) = x_g + \rho_L \sin(\Omega_c t - \phi) \quad x_g \equiv x_0 + \rho_L \sin \phi 
\)

\( y(t) = y_g + \rho_L \cos(\Omega_c t - \phi) \quad y_g \equiv y_0 - \rho_L \cos \phi \)

The guiding center position is \((x_g, y_g)\); the larmor (or gyro) radius is \( \rho_L = v_\perp / \Omega_c = m v_\perp / qB \); the larmor (or gyro) frequency is \( \Omega_c \)

### 4.1.3 Single Particle Drifts

In this section, \( R_c \) is the particle's radius of curvature in a magnetic field and is defined as \( \mathbf{b} \cdot \nabla \mathbf{b} = -R_c / R_c^2 \)

**\( E \times B \) drift** \( 8:149 \):

\[
\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}
\]

**\( \nabla B \) drift** \( 8:153 \):

\[
\mathbf{v}_{\nabla B} = \frac{m v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2}
\]

**Curvature drift** \( 8:159 \):

\[
\mathbf{v}_\kappa = \frac{m v_\parallel^2}{qB} \frac{R_c \times \mathbf{B}}{R_c^2 B}
\]

**Polarization drift** \( 8:162 \):

\[
\mathbf{v}_p = \frac{m}{qB} \hat{\mathbf{b}} \times \frac{d\mathbf{v}_E}{dt}
\]

Vacuum field only \( \rightarrow \nabla \times \mathbf{B} = 0 \) \( 8:160 \):

\[
\mathbf{v}_{\nabla B} + \mathbf{v}_\kappa = \frac{m}{qB} (v_\parallel^2 + \frac{v_\perp^2}{2}) \frac{R_c \times \mathbf{B}}{R_c^2 B}
\]

Particle drift velocity for a general force \( \mathbf{F} \) \( 8:153 \):

\[
\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2}
\]
4.2. BINARY COULOMB COLLISIONS

4.1.4 Magnetic Moment And Mirroring

In this section, \( i \) refers to the initial point, and \( f \) stand for the final, or mirror, point.

Magnetic moment (the first adiabatic invariant) \(^8:167\)

\[
\mu = \frac{mv_i^2(t)}{2B(t)} = \text{constant}
\]

Force on particle in magnetic fields where \( \nabla B/B \ll 1 \) \(^8:171\)

\[
F_{||} \approx -\mu \nabla_{||} B
\]

Velocity in terms of velocity space pitch angle \(^8:174\)

\[
v_{\perp i} = v_0 \sin \theta \quad v_{\parallel i} = v_0 \cos \theta
\]

Conservation of energy \(^8:174\)

\[
\frac{1}{2}m(v_{\perp i}^2 + v_{\parallel i}^2) = \frac{1}{2}mv_{\perp f}^2 = U_{\text{total}} = \text{constant}
\]

Mirroring condition \(^8:175\)

\[
\sin^2 \theta_c = \frac{U_{\perp i}}{U_{\text{total}}} = \frac{v_{\perp i}^2}{v_{\perp f}^2} = \frac{B_{\text{min}}}{B_{\text{max}}}
\]

Fraction of trapped particles (Maxwellian distribution) \(^8:176\)

\[
F_{\text{trapped}} = \frac{1}{n} \int_{-\pi}^{\pi} \sin \theta d\theta \int_{0}^{2\pi} d\phi \int_{0}^{\infty} F_{\text{Maxwellian}}(v)v^2 dv
\]

where \( n \) is the total number of particles in the distribution function

4.2 Binary Coulomb Collisions

\( r \) is the relative distance

\( v_1 \) and \( v_2 \) are particle velocities in the lab frame

\( V \) and \( v \) are the center of mass and relative velocities

\( v_0 \) and \( b_0 \) are the initial relative velocity and impact parameter

\( \chi \) is the scattering angle in the center of mass frame

\( \dot{x} \) is the time derivative of quantity \( x \)

Force between 2 charged particles

\[
F = -\nabla \left( \frac{q_1 q_2}{4\pi \epsilon_0 r} \right)
\]

Transformation to center of mass frame \(^8:186\)

\[
V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad v = v_1 - v_2
\]

\[
v_1 = V + \frac{m_2 v}{m_1 + m_2} \quad v_2 = V - \frac{m_1 v}{m_1 + m_2}
\]
Schematic of two body collision in the reduced mass frame.

Reduced mass \( m_\mu = \frac{m_1 m_2}{m_1 + m_2} \)

Conservation of energy \( E_0 = \frac{1}{2} m_\mu v_0^2 = \) constant

Conservation of angular momentum \( m_\mu \mathbf{r} \times \mathbf{v} = \mathbf{L}_0 = -m_\mu b v_0 = \) constant

Transformation to cylindrical coordinates
\[
\mathbf{r} = r \hat{r} \quad \mathbf{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}
\]

Ordinary differential equation for unknown \( r(t) \)
\[
\dot{r} = \mp v_0 \left( 1 - 2 \frac{b_{90}}{r} - \frac{b^2}{r^2} \right)^{1/2}
\]

Solution in terms of scattering angle \( \chi \)
\[
\tan \left( \frac{\chi}{2} \right) = \frac{b_{90}}{b} = \frac{q_1 q_2}{4 \pi \varepsilon_0 m_\mu v_0^2 b}
\]

Impact parameter for 90 degree collision \( b_{90} = \frac{q_1 q_2}{4 \pi \varepsilon_0 m_\mu v_0^2} \)

4.3 Single Particle Collisions with Plasma

\( a \) is an approaching test particle; \( t \) is a target plasma particle. \( Q_{at} \) are quantities depending on approaching particle \( a \) incident upon target particle \( t \).
4.3. SINGLE PARTICLE COLLISIONS WITH PLASMA

Total loss in test particle linear momentum \(8:192\)

\[
\frac{d}{dt}(m_a v_a) = -(\Delta m_a v_a) n_t \sigma v_a
\]
\[
= - \int (\Delta m_a v_a) f_a(v_t) |v_a - v_t| b \, db \, d\Omega \, d^3 v
\]

Definition of test particle collision frequency \(8:193\)

\[
\frac{d}{dt}(m_a v_a) = -\nu_{at}(m_a v_a)
\]

Test particle collision frequency \(8:193\)

\[
\nu_{at}(v_a) = \frac{1}{m_a v_a} \int (\Delta m_a v_a) f_t(v_t) |v_a - v_t| b \, db \, d\Omega \, d^3 v
\]

4.3.1 Collision Frequencies

Approximated expressions for frequencies hold only for \(v_e \sim v_{te} \gg v_{ti}\)

Electron-ion \(8:197\)

\[
\nu_{ei} = \frac{\left(\frac{e^4 n_i \ln \Lambda}{4\pi \epsilon_0^2 m_e m_\mu}\right)}{v_e^3 + 1.3 v_{te}^3} \approx \frac{e^4 n_i \ln \Lambda}{4\pi \epsilon_0^2 m_e^2 v_e^3}
\]

\[
\approx 8.06 \times 10^5 \frac{n_i \ln \Lambda}{v_e^3} \, [s^{-1}]
\]

Electron-electron \(8:197\)

\[
\nu_{ee} = \frac{\left(\frac{e^4 n_e \ln \Lambda}{2\pi \epsilon_0^2 m_e^2}\right)}{v_e^3 + 1.3 v_{te}^3} \, [s^{-1}]
\]

Ion-ion \(8:197\)

\[
\nu_{ii} = \frac{\left(\frac{e^4 n_i \ln \Lambda}{2\pi \epsilon_0^2 m_i^2}\right)}{v_i^3 + 1.3 v_{ti}^3} \, [s^{-1}]
\]

Ion-electron \(8:197\)

\[
\nu_{ie} = \frac{\left(\frac{e^4 n_e \ln \Lambda}{4\pi \epsilon_0^2 m_e m_i}\right)}{v_i^3 + 1.3 v_{te}^3} \, [s^{-1}]
\]

Collision frequency scalings \(8:197\)

\[
\nu_{ee} \sim \nu_{ei} \quad \nu_{ii} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ei} \quad \nu_{ie} \sim \left(\frac{m_e}{m_i}\right) \nu_{ei}
\]

The Coulomb logarithm \(8:194\)

\[
\Lambda \approx \frac{12 \pi^{3/2} T_e^{3/2}}{n_e^{1/2} e^3} = 4.9 \times 10^7 \frac{T_{kev}^{3/2}}{n_{20}^{1/2}}
\]

where \(\ln \Lambda \approx 15 - 20\) for most plasmas.
4.3.2 Collision Times

Electron-ion collision time \(\tau_{ei}\)

\[
\tau_{ei} = \frac{12\pi^{3/2}e_0^2m_e^{1/2}T_e^{3/2}}{\sqrt{2}n_iZ_i^2e^4\ln\Lambda}
\]

\[
= 1.09 \times 10^{16} \frac{T_e^{3/2}}{Z_i^2n_i\ln\Lambda} \text{ [s]}
\]

Ion-ion collision time \(\tau_{ii}\)

\[
\tau_{ii} = \frac{12\pi^{3/2}e_0^2m_i^{1/2}T_i^{3/2}}{21^{1/2}n_iZ_i^4e^4\ln\Lambda_i}
\]

\[
= 4.67 \times 10^{17} \left(\frac{m_i}{m_p}\right)^{1/2} \frac{T_i^{3/2}}{Z_i^4n_i\ln\Lambda_i} \text{ [s]}
\]

An ion collision time is sometimes defined to be \(\tau_i = \sqrt{2}\tau_{ii}\)

Ion-impurity collision time \(\tau_{iI}\)

\[
\tau_{iI} = \frac{12\pi^{3/2}m_i^{1/2}T_i^{3/2}e_0^2}{21^{1/2}n_IZ_i^2Z_i^2e^4\ln\Lambda_i}
\]

\[
= 4.67 \times 10^{17} \left(\frac{m_i}{m_p}\right)^{1/2} \frac{T_i^{3/2}}{Z_i^4Z_i^2n_i\ln\Lambda_i} \text{ [s]}
\]

Electron-to-ion energy transfer time

\(R_{ei} = \frac{3}{2}n (T_e - T_i) \left(\frac{m_i}{2m_e\tau_{ei}}\right)\)

Thermal equilibration frequency (rate of species a equilibrating to species b) \(\nu_{ab}\)

\[
\nu_{ab} = 1.8 \times 10^{-19} \frac{(m_am_b)^{1/2}Z_a^2Z_b^2n_b\ln\Lambda}{(m_aT_b + m_bT_a)^{3/2}} \text{ [s}^{-1}\]
\]

For ions and electrons with \(T_e \approx T_i = T\), \(\nu_{ei}n_e = \nu_{ie}n_i\)

\[
\nu_{ei} = 3.2 \times 10^{-15} \frac{Z^2\ln\Lambda}{(m_i/m_pT_i^{3/2})} \text{ [m}^3\text{s}^{-1}\]

4.4 Particle Beam Collisions with Plasmas

In this section, plasma density is written as \(n_{20} = n/10^{20}\)
4.4. PARTICLE BEAM COLLISIONS WITH PLASMAS

Beam-electron collision frequency $^{8:202–203}$

$$\nu_{be}(v_b) = \frac{Z_b^2 e^4 n_e \ln \Lambda}{4\pi \epsilon_0^2 m_e m_b} \frac{1}{v_b^3 + 1.3 v_{te}^3}$$

$$\approx \frac{1}{3(2\pi)^{3/2}} \frac{Z_b^2 e^4 m_e^{1/2} n_e \ln \Lambda}{\epsilon_0^2 m_b T_e^{3/2}} \left( v_{te}^3 \gg v_b^3 \right.)$$

$$= 100 \frac{n_{20}}{T_{keV}^{3/2}} \left[ \text{s}^{-1} \right] \left( \text{alpha heating only} \right)$$

Beam-ion collision frequency $^{8:203}$

$$\nu_{bi}(v_b) = \frac{Z_b^2 e^4 n_i \ln \Lambda}{4\pi \epsilon_0^2 m_i m_b} \frac{1}{v_b^3 + 1.3 v_{ti}^3}$$

$$\approx \frac{1}{4\pi} \frac{Z_b^2 e^4 n_i \ln \Lambda}{\epsilon_0^2 m_b v_b^3} \left( v_{ti}^3 \ll v_b^3 \right.)$$

$$= 0.94 \frac{n_{20}}{(E_{\text{beam}})^{3/2}} \left[ \text{s}^{-1} \right] \left( \text{alpha heating only} \right)$$

When $\nu_{be} = \nu_{bi}$, $v_b = v_{crit}$ and the beam changes the plasma particle it preferentially damps upon. For $v_b > v_{crit}$, the beam damps on plasma electrons; for $v_b < v_{crit}$, the beam damps on plasma ions. The critical beam energy is calculated as $E_c = 1/2m_{\text{beam}}v_{beam}^2 = 14.8(m_{\text{beam}}/m_i^{2/3})T_e$. For a 15 keV plasma, the critical energy corresponds to 660 keV.

A high energy beam entering a plasma has energy behavior (purely slowing down on electrons) $^{26:249}$

$$E_b = E_{b,0} \left[ e^{-3t/\tau_{se}} - \left( \frac{E_c}{E_{b,0}} \right)^{3/2} \left( 1 - e^{-3t/\tau_{se}} \right) \right]^{2/3}$$

where $E_{b,0}$ is the initial beam energy and $\tau_{se}$ is the slowing down time assuming $v_b < v_{te}$ $^{26:246}$

$$\tau_{se} = \frac{3(2\pi)^{3/2} \epsilon_0^2 m_b T_e^{3/2}}{m_c^{1/2} n e^4 \ln \Lambda}$$
Chapter 5

Plasma Parameters and Definitions

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

\( e \) is the elementary electric charge
\( Z \) is atomic (proton) number
\( m \) is the particle mass
\( e \) and \( i \) subscripts refer to electrons and ions, respectively
\( B \) is the magnetic field
\( n \) is the particle density; \( n_{20} = n/10^{20} \)
\( T \) is temperature; \( T_{keV} = T \) in units of kiloelectron-volts

5.1 Single Particle Parameters

Thermal speed\(^1\)

\[
v_{te} = \left(\frac{T_e}{m_e}\right)^{1/2} \quad v_{ti} = \left(\frac{T_i}{m_i}\right)^{1/2}
\]

Plasma frequencies [radians/s] \(^{8,135}\)

\[
\omega_{pe} = \left(\frac{n_e e^2}{m_e \epsilon_0}\right)^{1/2} \quad \omega_{pi} = \left(\frac{n_i (Ze)^2}{m_i \epsilon_0}\right)^{1/2}
\]

Cyclotron frequencies [radians/s] \(^{8,134}\)

\[
\Omega_e = \frac{eB}{m_e} \quad \Omega_i = \frac{ZeB}{m_i}
\]

\(^1\)Some references define thermal speed with an additional factor of \(\sqrt{2}\). The definition given here results in \(v_t\) being the standard deviation of a Gaussian distribution, which is the widely used convention in statistics and mathematical physics. In either case, it’s just a question of how the factor of “1/2” in the exponential term of the Gaussian distribution is treated.
Gyro radii \([m]\):

\[
\rho_{Le} = \frac{(2m_e T_e)^{1/2}}{eB} \quad \rho_{Li} = \frac{(2m_i T_i)^{1/2}}{ZeB}
\]

Single particle parameters as functions of magnetic field \(B\) [T], density \(n_{20} \text{ [m}^{-3}\)], and temperature \(T\) [keV] ³.

<table>
<thead>
<tr>
<th>Particle</th>
<th>Plasma frequencies [(\omega_f)]</th>
<th>Cyclotron frequencies [(\omega_f)]</th>
<th>Gyro radii [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>electron</td>
<td>(5.641 \times 10^{11} n_{20}^{1/2} \text{ rad/s})</td>
<td>(1.759 \times 10^{11} B \text{ rad/s})</td>
<td>(1.066 \times 10^{-4} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(89.779 n_{20}^{1/2} \text{ GHz})</td>
<td>28,000 (B) (\text{GHz})</td>
<td></td>
</tr>
<tr>
<td>proton</td>
<td>(1.316 \times 10^{10} n_{20}^{1/2} \text{ rad/s})</td>
<td>(9.579 \times 10^7 B \text{ rad/s})</td>
<td>(4.570 \times 10^{-3} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(2.094 n_{20}^{1/2} \text{ GHz})</td>
<td>15,241 (B) (\text{MHz})</td>
<td></td>
</tr>
<tr>
<td>deuteron</td>
<td>(9.312 \times 10^9 n_{20}^{1/2} \text{ rad/s})</td>
<td>(4.791 \times 10^7 B \text{ rad/s})</td>
<td>(6.461 \times 10^{-3} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(1.482 n_{20}^{1/2} \text{ GHz})</td>
<td>7,626 (B) (\text{MHz})</td>
<td></td>
</tr>
<tr>
<td>triton</td>
<td>(7.609 \times 10^9 n_{20}^{1/2} \text{ rad/s})</td>
<td>(3.200 \times 10^7 B \text{ rad/s})</td>
<td>(7.906 \times 10^{-3} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(1.211 n_{20}^{1/2} \text{ GHz})</td>
<td>5,092 (B) (\text{MHz})</td>
<td></td>
</tr>
<tr>
<td>helion</td>
<td>(7.610 \times 10^9 n_{20}^{1/2} \text{ rad/s})</td>
<td>(6.400 \times 10^7 B \text{ rad/s})</td>
<td>(3.952 \times 10^{-3} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(1.211 n_{20}^{1/2} \text{ GHz})</td>
<td>10,186 (B) (\text{MHz})</td>
<td></td>
</tr>
<tr>
<td>alpha</td>
<td>(6.605 \times 10^9 n_{20}^{1/2} \text{ rad/s})</td>
<td>(4.822 \times 10^7 B \text{ rad/s})</td>
<td>(4.554 \times 10^{-3} T_{\text{keV}}^{1/2} B)</td>
</tr>
<tr>
<td></td>
<td>(1.051 n_{20}^{1/2} \text{ GHz})</td>
<td>7,675 (B) (\text{MHz})</td>
<td></td>
</tr>
</tbody>
</table>

### 5.2 Plasma Parameters

Debye length ³:

\[
\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} = \frac{e^2 n_0}{\epsilon_0 T_e} + \frac{e^2 n_0}{\epsilon_0 T_i}
\]

\[
\lambda D_e \approx \left( \frac{\epsilon_0 T_e}{e^2 n_0} \right)^{1/2} = 2.35 \times 10^{-5} \left( \frac{T_{\text{keV}}}{n_{20}} \right)^{1/2} \text{ [m]}
\]

Debye-shield ion potential (spherical coordinates) ²⁶ ³⁷:

\[
V = \frac{e}{4\pi\epsilon_0 r} e^{-r/\lambda_D}
\]

Volume of a Debye sphere ³:

\[
V_D = \frac{4}{3}\pi\lambda_D^3
\]
5.3 **PLASMA SPEEDS**

Plasma parameter $^8:133$

$$
A_D = \nu_D n_0 = \frac{4 \pi}{3} \left( \frac{e_0 T_e}{e^2 n_0} \right)^{3/2} n_0 \approx 5.453 \times 10^6 \frac{T_{keV}^{3/2}}{n_0^{1/2}}
$$

Effective plasma charge $^8:56$

$$
Z_{\text{eff}} = \frac{\sum_{\text{all ions}} n_j Z_j^2}{\sum_{\text{all ions}} n_j Z_j} = \frac{1}{n_e} \sum_{\text{all ions}} n_j Z_j^2
$$

### 5.3 Plasma Speeds

In this section, $\rho_0$ is the mass density of the plasma, $p_0$ is the plasma pressure, and $\gamma$ is the adiabatic index.

**Alfvén speed** $^8:314$

$$
v_a = (B_0^2/\mu_0 \rho_0)^{1/2}
$$

**Sound speed** $^5:96,67$

$$
c_s \approx \left( \frac{Z T_e + \gamma T_i}{m_i} \right)^{1/2}
$$

where

- $\gamma = 1$ (isothermal flow)
- $\gamma = \frac{2 + N}{N}$ (N is the number of degrees of freedom)

**Plasma Mach number** $^5:298$

$$
M \equiv \frac{v_{\text{plasma}}}{c_s}
$$

### 5.4 Fundamentals of Maxwellian Plasmas

General Maxwellian velocity distribution function $^{21:64–65}$

$$
\mathcal{F}_M(v_x, v_y, v_z) = \mathcal{F}_M(\mathbf{v}) = C \exp \left( -\frac{b m}{2} \left[ (v_x - a_x)^2 + (v_y - a_y)^2 + (v_z - a_z)^2 \right] \right)
$$

where $a_x, a_y, a_z, b,$ and $c$ are constants. If $a_x = a_y = a_z = 0$ we have an (ordinary) Maxwellian; otherwise we have a drifting Maxwellian where the drift (or mean) velocity is $\mathbf{v}_{dr} = (a_x, a_y, a_z)$.

**Ordinary Maxwellian velocity distribution function for a plasma** $^{21:64–65}$

$$
\mathcal{F}_M(\mathbf{v}) = n \left( \frac{m}{2 \pi T} \right)^{3/2} \exp \left( -\frac{m}{2T} \left( v_x^2 + v_y^2 + v_z^2 \right) \right)
$$
Definition of temperature in a Maxwellian plasma 

\[ \frac{3}{2} nT \equiv n \left( \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) \right) \equiv \int \mathcal{F}_M(v) \left( \frac{1}{2} (v_x^2 + v_y^2 + v_z^2) \right) dv \]

Total number density of particles in a Maxwellian plasma

\[ n = \int_{\text{all velocity space}} \mathcal{F}_M(v) dv = 4\pi \int_0^\infty w^2 \mathcal{F}_M(w) dw \]

where we have transformed to spherical coordinates such that \( dv = w^2 \sin \theta dw d\theta d\phi \) and \( w = (v_x^2 + v_y^2 + v_z^2)^{1/2} \).

Average (thermal) particle speed in a Maxwellian plasma

\[ \bar{c} \equiv \frac{1}{n} \int_0^\infty w \mathcal{F}_M(w) dw = \left( \frac{8T}{\pi m} \right)^{1/2} \]

Thermal particle flux in a single dimension \( x \) for a Maxwellian plasma

\[ \Gamma \equiv \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty v_x \mathcal{F}_M(v) dv_x dv_y dv_z = \frac{1}{4} n\bar{c} \]

5.5 Definition of a Magnetic Fusion Plasma

A fusion plasma is defined as an electrically conducting ionized gas that is dominated by collective effects and that magnetically confines its composing particles. If \( L \) is the macroscopic length scale of the plasma, \( \omega_{\text{transit}} \) is 1 over the time required for a particle to cross the plasma, and \( v_T \) is the thermal particle velocity, the criteria to be a fusion plasma are:

<table>
<thead>
<tr>
<th>Required condition</th>
<th>Physical consequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_D \ll L )</td>
<td>Shielding of DC electric fields</td>
</tr>
<tr>
<td>( \omega_{pe} \gg \omega_{\text{transit}} = v_{te}/L )</td>
<td>Shielding of AC electric fields</td>
</tr>
<tr>
<td>( \Lambda_D \gg 1 )</td>
<td>Collective effects dominate</td>
</tr>
<tr>
<td>( \rho_L \ll L )</td>
<td>Magnetic confinement of particle orbits</td>
</tr>
<tr>
<td>( \Omega_i \gg v_{ti}/L )</td>
<td>Particle gyro orbits dominate free streaming</td>
</tr>
</tbody>
</table>
5.6 Fundamental Plasma Definitions

5.6.1 Resistivity

Plasma resistivity of an unmagnetized plasma

\[ \eta = \frac{m_e}{n_e e^2 \tau_e} \approx 5.2 \times 10^{-5} \frac{Z_{\text{eff}} \ln \Lambda}{T_e^{3/2}} \]  [Ω · m]

Spitzer resistivity of a singly charged unmagnetized plasma

\[ \eta_s = 0.51 \frac{m_e}{n_e e^2 \tau_e} = 0.51 \frac{m_e^{1/2} e^2 \ln \Lambda}{3 \epsilon_0^2 (2 \pi T_e)^{3/2}} \]
\[ \approx 1.65 \times 10^{-9} \frac{\ln \Lambda}{T_e^{3/2}} \]  [Ω · m]

Spitzer resistivity of a singly charged magnetized plasma

\[ \eta_{s||} \text{ to } B = \eta_s \]
\[ \eta_{s\perp} \text{ to } B = 1.96 \eta_s \]

Spitzer resistivity of a plasma with impurities

\[ \eta_s = Z_{\text{eff}} \eta_s \]

Spitzer resistivity of a pure non-hydrogenic plasma of charge \( Z \)

\[ \eta(Z) = N(Z) Z \eta_s \]

where
\[ \begin{cases} 
N = 0.85 \text{ for } Z = 2 \\
N = 0.74 \text{ for } Z = 4 
\end{cases} \]

5.6.2 Runaway Electrons

Volumetric runaway electron production rate

\[ R = \frac{2}{\sqrt{\pi}} \frac{n}{\tau_{\text{se}}} \left( \frac{E}{E_D} \right)^{1/2} \exp \left[ -\frac{E_D}{4E} \left( \frac{2 E_D}{E} \right)^{1/2} \right] \]

where the Driecer electric field, \( E_D \) is

\[ E_D = \frac{n e^3 \ln \Lambda}{4 \pi \epsilon_0^2 m_e v_{te}^2} \]
\[ \approx 4.582 \times 10^6 \frac{n \ln \Lambda}{v_{te}^2} \]  [V/m]

and the electron slowing down time, \( \tau_{\text{se}} \), for \( v_e \gg v_{te} \) is

\[ \tau_{\text{se}} = \frac{4 \pi \epsilon_0^2 m_e^2 v_e^3}{n e^4 \ln \Lambda} \]
\[ \approx 1.241 \times 10^{-6} \frac{v_e^3}{n \ln \Lambda} \]  [s]

Relativistic runaway electron limit (Connor-Hastie limit)

\[ E < \frac{n e^3 \ln \Lambda}{4 \pi \epsilon_0^2 m_e c^2} \]
\[ \approx 4.645 \times 10^{-53} \frac{n \ln \Lambda}{m_e} \]
Chapter 6

Plasma Models

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

\( \mathbf{u} \) is the plasma flow velocity
\( \mathbf{v} \) is the particle velocity vector
\( \mathbf{a} \) is the particle acceleration vector
\( m \) is the particle mass
\( \alpha \) is a general particle
\( n_\alpha \) is the number density of particle \( \alpha \)
\( \rho \) is the plasma mass density
\( p \) is the plasma pressure
\( a \) and \( R_0 \) are the minor and major radius of a toroidal plasma
\( \kappa \) is the plasma elongation
\( e \) is the fundamental charge unit
\( i \) and \( e \) subscripts refer to the ions and electrons, respectively

6.1 Kinetic

The kinetic Vlasov equation\(^7\)\(^{10}\)

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \nabla_v f = C(f)
\]

where \( C(f) \) is the integro-differential collision operator.

The collisionless kinetic Vlasov equation

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0
\]

where in a Maxwellian distribution\(^8\)\(^{46}\)

\[
f = \frac{1}{(\pi v_t^*)^{3/2}} \exp \left(-v^2/2v_t^2\right)
\]
6.2 Two Fluids

Continuity equation \(^7:15\)

\[
\left( \frac{dn_\alpha}{dt} \right)_\alpha + n_\alpha \nabla \cdot \mathbf{v}_\alpha = 0
\]

Momentum equation \(^7:15\)

\[
n_\alpha m_\alpha \left( \frac{dv_\alpha}{dt} \right)_\alpha - q_\alpha n_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) + \nabla \cdot \mathbf{P}_\alpha = R_\alpha
\]

Energy equation \(^7:15\)

\[
\frac{3}{2} n_\alpha \left( \frac{dT_\alpha}{dt} \right)_\alpha + \mathbf{P}_\alpha : \nabla \mathbf{v}_\alpha + \nabla \cdot \mathbf{h}_\alpha = Q_\alpha
\]

Maxwell’s equations \(^7:15\)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 (Z_{\text{en}_i} \mathbf{v}_i - e_{\text{en}_e} \mathbf{v}_e) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0} (n_i - n_e)
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

where

Convective derivative \(^7:14\)

\[
\left( \frac{d}{dt} \right)_\alpha = \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla
\]

Heat generated by unlike collision \(^7:13\)

\[
Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta} d\mathbf{w}
\]

Mean momentum transfer from unlike particles \(^7:14\)

\[
R_\alpha \equiv \int m_\alpha \mathbf{w} C_{\alpha\beta} d\mathbf{w}
\]

Heat flux due to random motion \(^7:14\)

\[
\mathbf{h}_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \langle w^2 \mathbf{w} \rangle
\]
6.2. TWO FLUIDS

Temperature $T_\alpha \equiv p_\alpha / n_\alpha$

Anisotropic part of pressure tensor $\Pi_\alpha \equiv \mathbf{P}_\alpha - p_\alpha \mathbf{I}$

Pressure tensor $\mathbf{P}_\alpha \equiv n_\alpha m_\alpha \langle \mathbf{w} \mathbf{w} \rangle$

Scalar pressure $p_\alpha \equiv \frac{1}{3} n_\alpha m_\alpha \langle w^2 \rangle$

Collision operator $C(f) = \sum_\beta C_{\alpha \beta}$

Macroscopic number density

$n_\alpha (r, t) \equiv \int f_\alpha d\mathbf{u}$

Macroscopic fluid velocity

$v_\alpha (r, t) \equiv \frac{1}{n_\alpha} \int \mathbf{u} f_\alpha d\mathbf{u}$

Velocity representing thermal motion of the particles

$\mathbf{w} = \mathbf{u} - v_\alpha (r, t)$

Relations involving the collision operators $C_{ij}$

(a) $\int C_{ee} d\mathbf{u} = \int C_{ii} d\mathbf{u} = \int C_{ei} d\mathbf{u} = \int C_{ie} d\mathbf{u} = 0$

(b) $\int m_e u C_{ee} d\mathbf{u} = \int m_i u C_{ii} d\mathbf{u} = 0$

(c) $\int \frac{1}{2} m_e u^2 C_{ee} d\mathbf{u} = \int \frac{1}{2} m_i u^2 C_{ii} d\mathbf{u} = 0$

(d) $\int (m_e u C_{ei} + m_i u C_{ie}) d\mathbf{u} = 0$

(e) $\int \frac{1}{2} (m_e u^2 C_{ei} + m_i u^2 C_{ie}) d\mathbf{u} = 0$
6.3 One Fluid

Taking the two fluid equation, we assume $m_e \to 0$ and $n_i = n_e \equiv n$ so that

\[ \rho = m_i n \]
\[ \mathbf{v} = \mathbf{v}_i \]
\[ \mathbf{v}_e = \mathbf{v} - \mathbf{J}/en \]

Additional simplifications $p = nT = p_e + p_i$ and $T = T_e + T_i$ lead to the one-fluid equations

Continuity of mass equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \]

Continuity of charge equation

\[ \nabla \cdot \mathbf{J} = 0 \]

Momentum equation

\[ \rho \frac{d\mathbf{v}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\mathbf{\Pi}_i + \mathbf{\Pi}_e) \]

Force balance

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en} \left( \mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \mathbf{\Pi}_e \mathbf{R}_e \right) \]

Equations of state

\[ \frac{d}{dt} \left( \frac{p_i}{\rho^\gamma} \right) = \frac{2}{3 \rho^\gamma} \left( Q_i - \nabla \cdot \mathbf{h}_i - \mathbf{\Pi}_i : \nabla \mathbf{v} \right) \]

\[ \frac{d}{dt} \left( \frac{p_e}{\rho^\gamma} \right) = \frac{2}{3 \rho^\gamma} \left[ Q_e - \nabla \cdot \mathbf{h}_e - \mathbf{\Pi}_e : \nabla \left( \frac{\mathbf{v} - \mathbf{J}}{en} \right) \right] + \frac{1}{en} \mathbf{J} \cdot \nabla \left( \frac{p_e}{\rho^\gamma} \right) \]

Maxwell’s equation (low frequency limit)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
6.4 Magnetohydrodynamics (MHD)

MHD scalings \(^8:247\)

- length: \(a >> \rho_{Li} >> [\rho_{Le} \sim \lambda_{De}]\)
- frequency: \(v_{ce} << v_{ti}/a << \omega_{ci} << [\omega_{ce} \sim \omega_{pe}]\)
- velocity: \(v_{ti} \sim v_a << v_{te} << c\)

where \(a\) is the tokamak minor radius.

The MHD equations

**Continuity of mass** \(^8:252\)

\[
\frac{dp}{dt} + \rho \nabla \cdot \mathbf{v} = 0
\]

**Momentum equation** \(^8:252\)

\[
\frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p
\]

**Ohm’s law** \(^8:252\)

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \text{ (ideal MHD)}
\]

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta_{||} \mathbf{J} \text{ (resistive MHD)}
\]

**Equation of state** \(^8:252\)

\[
\frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0
\]

**Maxwell’s equation (low frequency limit)** \(^8:252\)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

where \(\eta_{||}\) is the parallel resistivity.

6.4.1 Frozen-in Magnetic Field

In ideal MHD, it is found that \(^8:300\)

\[
\frac{d\Phi}{dt} = 0
\]

where \(\Phi = \int \mathbf{B} \cdot d\mathbf{S}\) is the magnetic flux.
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CHAPTER 6. PLASMA MODELS

MHD relations for linear plasma devices

<table>
<thead>
<tr>
<th>Case</th>
<th>Current Relation</th>
<th>Equilibrium Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ-pincho 8:265</td>
<td>( \mu_0 J_\theta = -\frac{dB_\theta}{dr} )</td>
<td>( \frac{d}{dr} \left( p + \frac{B^2_\theta}{2\mu_0} \right) = 0 )</td>
</tr>
<tr>
<td>z-pincho 8:267</td>
<td>( \mu_0 J_z = \frac{1}{r} \frac{d(rB_\theta)}{dr} )</td>
<td>( \frac{d}{dr} \left( p + \frac{B^2_\theta}{2\mu_0} + \frac{B^2_\phi}{\mu_0 r} \right) = 0 )</td>
</tr>
<tr>
<td>screw pincho 8:269</td>
<td>( \mu_0 J = -\frac{dB_\theta}{dr} \hat{\theta} + \frac{1}{r} \frac{d(rB_\phi)}{dr} \hat{z} )</td>
<td>( \frac{d}{dr} \left( p + \frac{B^2_\phi}{2\mu_0} + \frac{B^2_\theta}{2\mu_0} \right) + \frac{B^2_\phi}{\mu_0 r} = 0 )</td>
</tr>
</tbody>
</table>

6.5 MHD Equilibria

The generalized MHD equilibrium equations 8:261

\[ \mathbf{J} \times \mathbf{B} = \nabla p \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \nabla \cdot \mathbf{B} = 0 \]

6.6 Grad-Shafranov Equation

The Grad-Shafranov equation is the solution of the ideal MHD equations in two dimensions. In this section, \( A_\phi \) is the toroidal component of the vector potential. 7:110–111

\[ \Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{F}{\psi} \frac{dF}{d\psi} \]

where

\[ \mathbf{B} = \frac{1}{R} \nabla \psi \times \hat{\phi} + \frac{F}{R} \hat{\phi} \]

\[ \mu_0 \mathbf{J} = \frac{1}{R} \frac{dF}{d\psi} \nabla \psi \times \hat{\phi} - \frac{1}{R} \Delta^* \psi \hat{\phi} \]

where

\[ \Delta^* \psi \equiv R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} \]

\[ \psi = \frac{\psi_p}{2\pi} = RA_\phi = \frac{1}{2\pi} \int \mathbf{B}_p \cdot d\mathbf{A} \]

\[ F(\psi) = RB_\phi \]

6.7 MHD Stability

This section deals with how a plasma behaves when it is perturbed slightly from equilibrium. Therefore, most physical parameters have an equilibrium value (subscript 0) with an added perturbed value (subscript 1). We also define the displacement vector \( \xi = \int \tilde{v}_1 \). 8:311
6.7. MHD STABILITY

Full Solutions to the Grad-Shafranov Equation

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>High $\beta$ Noncircular Tokamak $^7$: 148</td>
<td>$\frac{4}{7} r^2 + \frac{C}{8} r^3 \cos(\theta) + \sum_{m=0}^{\infty} H_m r^m \cos(m\theta)$</td>
</tr>
<tr>
<td>Spherical Tokamak $^7$: 163</td>
<td>$-\frac{A}{2} Z^2 + \frac{C}{8} R^4 + c_1 + c_2 R^2 + c_3 \left( R^4 - 4R^2 Z^2 \right)$</td>
</tr>
</tbody>
</table>

\[ \tilde{\rho}_1 = -\nabla \cdot (\rho_0 \tilde{\xi}) \]
\[ \tilde{p}_1 = -\tilde{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \tilde{\xi} \]
\[ \tilde{Q} \equiv \tilde{B}_1 = \nabla \times (\tilde{\xi} \times B_0) \]
\[ \rho \frac{\partial^2 \tilde{\xi}}{\partial t^2} = F(\tilde{\xi}) \]
\[ F(\tilde{\xi}) = J \times \tilde{B}_1 + \tilde{J}_1 \times B - \nabla \tilde{p}_1 \]
\[ F(\tilde{\xi}) = \frac{1}{\mu_0} (\nabla \times B) \times \tilde{Q} + \frac{1}{\mu_0} (\nabla \times Q) \times B + \nabla (\tilde{\xi} \cdot \nabla p + \gamma p \nabla \cdot \tilde{\xi}) \]

Then assuming all pertubed quantities $Q_1 = Q_1 \exp \left[ -i (\omega t - k \cdot r) \right]$, we can find expressions for the first order terms by letting $\frac{\partial}{\partial t} \rightarrow \omega$ and $\nabla \rightarrow k$.

6.7.1 Variational Formulation

A different way of looking at the stability problem is given by $^7$: 250

\[ \omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)} \]

where

\[ \delta W(\xi^*, \xi) = -\frac{1}{2} \int \xi^* \cdot F(\xi) dr \]
\[ = -\frac{1}{2} \int \xi^* \cdot \left[ \frac{1}{\mu_0} (\nabla \times Q) \times B + \frac{1}{\mu_0} (\nabla \times B) \times Q \right. \]
\[ \quad \left. + \nabla (\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) \right] dr \]
\[ K(\xi^*, \xi) = \frac{1}{2} \int \rho |\xi|^2 dr \]

An equilibrium is stable if $\delta W \geq 0$. The energy principle can be evaluated simply in two cases: a conducting wall directly in contact to the plasma $(n \cdot \tilde{\xi}_{wall} = 0)$ or with a vacuum region next to the plasma. A vacuum region is more realistic than an adjacent conducting wall, so the variational principle becomes $\delta W = \delta W_F + \delta W_S + \delta W_V$, where $F$, $S$, $V$ refer to fluid, surface, and vacuum, respectively. $^7$: 261
\[ \delta W_F = \frac{1}{2} \int_F dr \left[ \frac{|Q|^2}{\mu_0} - \xi_\perp \cdot (J \times Q) + \gamma p |\nabla \cdot \xi|^2 + (\xi_\perp \cdot \nabla p) \nabla \cdot \xi_\perp \right] \]

\[ \delta W_S = \frac{1}{2} \int_S dS |n \cdot \xi_\perp|^2 n \cdot \nabla \left( p + \frac{B^2}{2\mu_0} \right) \]

\[ \delta W_V = \frac{1}{2} \int_V dr \frac{|\hat{B}_1|^2}{\mu_0} \]

\[ n \hat{B}_1|_{r_w} = 0 \]

\[ n \hat{B}_1|_{r_w} = n \cdot \nabla \times \left( \xi_\perp \times \hat{B} \right)|_{r_p} \]

where \(|A|\) refers to the jump in A from the vacuum to the plasma.

### 6.8 Stability of the Screw Pinch

The general screw pinch stability with a vacuum region is given by two different expressions. For internal modes,\(^7\): 293

\[ \delta W = \frac{2\pi^2 R_0}{\mu_0} \int_0^a \left( f \xi'^2 + g \xi^2 \right) dr \]

with \(\xi(a) = 0\). For external modes,\(^7\): 293

\[ \delta W = \frac{2\pi^2 R_0}{\mu_0} \left\{ \int_0^a \left( f \xi'^2 + g \xi^2 \right) dr + \left[ \left( kr B_z - m B_\theta \right) \frac{k r^2 F^2}{k^2_0} \right] \xi_a^2 \right\} \]

where \(\xi_a\) is arbitrary and

\[ f = \frac{r F^2}{k^2_0} \]

\[ g = \frac{k^2}{k^2_0} (\mu_0 p)' + \left( \frac{k^2 r^2}{k^2_0 r^2} - \frac{1}{k^2_0 r^2} \right) r F^2 + 2 \frac{k^2}{r k^4_0} \left( k B_z - \frac{m B_\theta}{r} \right) F \]

\[ \Lambda = - \frac{|m| K_a}{ka K'_a} \left[ 1 - \frac{1}{(K'_a)^2} \right] \left[ 1 - \frac{1}{(K'_a)^2} \right] \]

\[ F = k \cdot B = \frac{m B_\theta}{r} + k B_z \]
6.8. STABILITY OF THE SCREW PINCH

6.8.1 Suydam’s Criterion

For stability in a screw pinch \(^7\):\(^{298}\)

\[
\frac{\tau B_z^2}{\mu_0} \left( \frac{q'}{q} \right)^2 + 8p' > 0
\]
Chapter 7

Transport

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

\( e \) is the elementary electric charge  
\( Z \) is the atomic (proton) number  
\( m \) is the particle mass  
\( n \) is the number density; \( n_{20} = n/10^{20} \)  
\( \mathbf{v} \) is the particle velocity vector  
\( \rho \) is the plasma mass density  
\( T \) is the plasma temperature  
\( \chi \) is the thermal diffusivity  
\( q \) is the heat flux  
\( \| \) and \( \perp \) indicate parallel and perpendicular to \( \mathbf{B} \)  
\( a \) and \( R_0 \) are the minor and major radii of a toroidal plasma  
\( S \) is a general source term

7.1 Classical Transport

7.1.1 Diffusion Equations in a 1D Cylindrical Plasma

If the following approximations are made to the MHD single fluid equations 8:451–453

(a) neglect inertial terms: \( \rho \frac{\partial \mathbf{v}}{\partial t} = 0 \)

(b) split resistivity into components: \( \eta \mathbf{J} \rightarrow \eta_\perp \mathbf{J}_\perp + \eta_\parallel \mathbf{J}_\parallel \)

(c) equate electron and ion temperature: \( T_e \approx T_i \equiv T \)

(d) introduce the low-\( \beta \) tokamak expansion: \( B_z(r,t) = B_0 + \delta B_z(r,t) \)  
    where \( B_0 \) is a constant and \( \delta B_z \ll 1 \)

then a short calculation leads a set of diffusion-like transport equations
Particles\textsuperscript{8:453}
\[
\frac{\partial n}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r D_n \left( \frac{\partial n}{\partial r} + \frac{n}{T} \frac{\partial T}{\partial r} + \frac{2\eta_\parallel}{\beta_p} \frac{n}{r B_\theta} \frac{\partial (r B_\theta)}{\partial r} \right) \right]
\]
\[D_n = \frac{2nT\eta_\perp}{B_0^2}\]
where the poloidal beta is \(\beta_p = 4\mu_0 nT/B_\theta^2 \sim 1\)

Temperature\textsuperscript{8:453}
\[
3n \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r n \chi \frac{\partial T}{\partial r} \right) + S
\]

Magnetic field\textsuperscript{8:453}
\[
\frac{\partial (r B_\theta)}{\partial t} = \frac{r}{\partial r} \left( \frac{D_B}{r} \frac{\partial (r B_\theta)}{\partial r} \right)
\]
\[D_B = \frac{\eta_\parallel}{\mu_0}\]

7.1.2 Classical Particle Diffusion Coefficients

Classical particle diffusion results from coulomb collisions. No shift in the center of mass occurs for like-particle collisions; therefore, \(D = 0\) for like-particles and only unlike-particle collisions lead to particle diffusion, implying \(D_n^{\text{electrons}} = D_n^{\text{ions}}\).

Net momentum electron-ion exchange collision frequency\textsuperscript{8:217}
\[
\bar{\nu}_{ei} = \sqrt{\frac{2}{\pi}} \frac{\omega_{pe}}{\Lambda} \ln \Lambda \approx 1.8 \times 10^5 \frac{n_{20}}{T_{3/2}^{3/2}} \text{keV}
\]

Random walk diffusion coefficient\textsuperscript{8:462}
\[
D_{rw} = 4 \bar{\nu}_{ei} m_e T_e e^2 B_0^2 \sim \frac{T_e}{\tau_{ei}}
\]

Fluid model diffusion coefficient\textsuperscript{8:463}
\[
D_{fm} = \frac{2nT\eta_\perp}{B_0^2} = 2 \bar{\nu}_{ei} m_e T_e e^2 B_0^2
\]

Braginskii diffusion coefficient\textsuperscript{8:463}
\[
D_{\text{Brag}} = 2.0 \times 10^{-3} \frac{n_{20}}{B_0^2 T_{4/2}^{1/2}} \text{[m}^2/\text{s]}\]

7.1.3 The Collision Operator

In this section, Einstein summation notation \((A_a B_a = \sum_a A_a B_a)\) is used, and \(u \equiv v - v'\), and \(d^3v'\) is a differential volume element in velocity space. The Fokker-Planck form of the collision operator\textsuperscript{11:29}
\[
C_{ab}(f_a, f_b) = \frac{\partial}{\partial v_k} \left[ A_k^{ab} f_a + \frac{\partial}{\partial v_l} (D_{kl}^{ab} f_a) \right]
\]
where \(^{11:28-29}\)

\[ A^{ab}_k \equiv \left( 1 + \frac{m_a}{m_b} \right) L^{ab} \frac{\partial \phi_b}{\partial v_k} \]

\[ D^{ab}_{kl} \equiv -L^{ab} \frac{\partial^2 \psi_b}{\partial v_k \partial v_l} \]

\[ L^{ab} \equiv \left( \frac{Z_a Z_b e^2}{m_a \epsilon_0} \right) \ln \Lambda \]

\[ \phi_b(v) \equiv -\frac{1}{4\pi} \int \frac{1}{u} f_b(v') d^3 v' \]

\[ \psi_b(v) \equiv -\frac{1}{8\pi} \int u f_b(v') d^3 v' \]

Rosenbluth form of the collision operator (general) \(^{11:30}\)

\[ C_{ab}(f_a, f_b) = -\frac{Z_a^2 Z_b^2 e^4 \ln \Lambda}{8\pi \epsilon_0^2 m_a} \int U_{kl} \left[ \frac{f_a(v) \partial f_b(v')}{m_b} \frac{\partial f_a(v)}{\partial v_l} - \frac{f_b(v') \partial f_a(v)}{m_a} \frac{\partial f_b(v')}{\partial v_l} \right] d^3 v' \]

where \(^{11:29}\)

\[ U_{kl} \equiv \frac{u^2 \delta_{kl} - u_k u_l}{u^3} \]

Rosenbluth form of the collision operator (Maxwellian plasma) \(^{11:37}\)

\[ C_{ab}(f_a, f_{b0}) = \nu_{ab}^D L(f_a) + \frac{1}{v^2} \frac{\partial}{\partial v} \left[ v^3 \left( \frac{m_a}{m_a + m_b} \nu_s^{ab} f_a + \frac{1}{2} \nu_s^{ab} v^2 \frac{\partial f_a}{\partial v} \right) \right] \]

where \( x_\alpha = v/v_{Tha} \) and \(^{11:38}\)

\[ L(f_a) \equiv \frac{1}{2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f_a}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial f_a}{\partial \phi} \right] \]

\[ G(x) \equiv \frac{\phi(x) - x \phi'(x)}{2x^2} = \begin{cases} \frac{2x}{3\sqrt{\pi}} & x \to 0 \\ \frac{1}{2x^2} & x \to \infty \end{cases} \]

\[ \nu_D^{ab}(v) = \nu_{ab} \frac{\text{erf}(x_b) - G(x_b)}{x_b^3} \]

\[ \nu_s^{ab}(v) = \nu_{ab} \frac{2T_a}{T_b} \left( 1 + \frac{m_b}{m_a} \right) \frac{G(x_b)}{x_a} \]
\[ \nu_{ab}^0(v) = 2\dot{\nu}_{ab}\frac{G(x_b)}{x_a^3} \]

\[ \dot{\nu}_{ab} = \frac{n_bZ_a^2Z_b^2e^4\ln A}{4\pi\epsilon_0^2m_a^2v_{Tha}^3} \]

Krook collision operator 11:84

\[ C(f) = \nu(f_0 - f) \]

where \( \nu \) is chosen as some characteristic collision time and \( f_0 \) is a base distribution frequency, which is often chosen to be a Maxwellian.

### 7.1.4 Classical Thermal Diffusivities

Random walk thermal diffusivities 8:464

\[ \chi_i = \frac{1}{4\Omega_i^2\tau_{ii}} \sim \frac{r_{Li}^2}{\tau_{ii}} \]

\[ \chi_e = \frac{1}{4\Omega_e^2\tau_{ee}} \sim \frac{r_{Le}^2}{\tau_{ee}} \]

Braginskii Thermal Diffusivity Coefficients (50%-50% D-T plasma) 8:465

\[ \chi_i = 0.10\frac{n_{20}}{B_0^2T_k^{1/2}} \text{ [m}^2/\text{s]} \]

\[ \chi_e = 4.8 \times 10^{-3}\frac{n_{20}}{B_0^2T_{keV}^{1/2}} \text{ [m}^2/\text{s]} \]

### 7.2 Neoclassical Transport

The following formulae are only valid in the so-called “banana regime” of tokamak confinement devices, where a significant fraction of confined particles undergo magnetic mirroring due to inhomogeneity of the magnetic fields.

#### 7.2.1 Passing Particles

Half-transit time 8:480

\[ \tau_{1/2} = \frac{1}{v_\parallel} = \frac{\pi R_0 q}{v_\parallel} \]

Drift Velocity 8:481

\[ v_D = \frac{m_i}{eB} \left( v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{R_e \times B}{R_e^2 B} \approx \frac{1}{\Omega_iR_0} \left( v_\parallel^2 + \frac{v_\perp^2}{2} \right) \left( \hat{r}\sin\theta + \hat{\theta}\cos\theta \right) \]
7.2. NEOCLASSICAL TRANSPORT

7.2.2 Trapped Particles

In this section, $q$ is the tokamak safety factor.

The minimum (inboard) and maximum (outboard) magnetic fields

$$B_{\text{min}} = B_0 \frac{R_0}{R_0 + a} \quad B_{\text{max}} = B_0 \frac{R_0}{R_0 - a}$$

Trapped particle condition

$$\frac{v_\parallel^2}{v^2} < 1 - \frac{B_{\text{min}}}{B_{\text{max}}} = 1 - \frac{R_0 - a}{R_0 + a} \approx \frac{2a}{R_0}$$

Fraction of trapped particles (maxwellian distribution $F_M$)

$$F_{\text{trapped}} = \frac{1}{n} \int_{\frac{\pi - \theta_c}{2}}^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} F_M(v)v^2 dv = \cos \theta_c \approx \left(\frac{2a}{R_0}\right)^{1/2}$$

Half-bounce time

$$\tau_{1/2} \approx \frac{l}{v_\parallel} \approx \frac{2l}{v_\parallel} \approx \frac{2\pi R_0 q}{v_\parallel}$$

Full-bounce frequency

$$\omega_B = \frac{v_\parallel}{2R_0 q}$$

Mean square step size (random walk model)

$$(\Delta l)^2 = \langle (\Delta r)^2 \rangle = 4 \frac{|v_D|^2}{\omega_B^2} \langle \cos^2 \theta_0 \rangle = 2 \frac{q^2 v^4}{\Omega_i^2 v_\parallel^2}$$

Mean step size (averaged over velocity)

$$(\Delta l)^2 \approx 3 \left( q^2 \frac{R_0}{a} \right) \frac{v_{\parallel i}^2}{\Omega_i^2} \sim \left( q^2 \frac{R_0}{a} \right) \rho_{Li}^2$$

where it has been assumed that

$$v^2 \sim \frac{3T}{m} = \frac{3}{2} v_{\parallel i}^2 \quad v_\parallel^2 \approx \frac{a}{R_0} v^2 \sim \frac{a}{2R_0} 3v_{\parallel i}^2$$

7.2.3 Trapped Particle Neoclassical Transport Coefficients

Random walk model neoclassical diffusion coefficient

$$D_{n}^{NC} = f \frac{(\Delta r)^2}{\tau_{\text{eff}}} = 5.2 q^2 \left( \frac{R_0}{a} \right)^{3/2} \left( \frac{2m_e T_e}{e^2 B_0^2 \tau_{ei}} \right) = 5.2 q^2 \left( \frac{R_0}{a} \right)^{3/2} D_{n}^{CL} \quad [\text{m}^2/\text{s}]$$

Neoclassical diffusion coefficient (Rosenbluth, Hazeltine, and Hinton)

$$D_{n}^{NC} = 2.2 q^2 \left( \frac{R_0}{a} \right)^{3/2} D_{n}^{CL}$$
Thermal diffusivities (Rosenbluth, Hazeltine, and Hinton) \(^8_{:489}\)

\[
\chi_{CN}^{i} = 0.68q^2 \left( \frac{R_0}{a} \right)^{3/2} \chi_{CL}^i = 0.068q^2 \left( \frac{R_0}{a} \right)^{3/2} \left( \frac{n_{20}}{B_0^2 T_{keV}^{1/2}} \right) \quad [m^2/s]
\]

\[
\chi_{CN}^{e} = 0.89q^2 \left( \frac{R_0}{a} \right)^{3/2} \chi_{CL}^e = 4.3 \times 10^{-3} q^2 \left( \frac{R_0}{a} \right)^{3/2} \left( \frac{n_{20}}{B_0^2 T_{keV}^{1/2}} \right) \quad [m^2/s]
\]

### 7.2.4 Transport regime criteria

#### Definition of collisionality \(^{11}_{:149}\)

\[\nu^* \equiv \nu q R/v_t e^{3/2}\]

The banana regime \(^{11}_{:149}\)

\[\nu q R/v_T \ll e^{3/2}\]

The plateau regime \(^{11}_{:149}\)

\[e^{3/2} \ll \nu q R/v_T \ll 1\]

The Pfirsch-Schluter regime \(^{11}_{:149}\)

\[\nu q R/v_T \gg 1\]
Chapter 8

Plasma Waves

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

\( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, respectively
\( \mathbf{b} \) is a unit vector in the direction of \( \mathbf{B} \)
\( || \) and \( \perp \) indicate parallel and perpendicular to \( \mathbf{b} \)
\( \mathbf{k} \) is the wave vector
\( \omega_{pi} \) is the plasma frequency for particle \( i \)
\( \Omega_i \) is the cyclotron frequency for particle \( i \)
\( \tilde{X} \) implies that \( X \) can be a complex number

8.1 Cold Plasma Electromagnetic Waves

Starting from Maxwell’s equation, linearize \( Q = \tilde{Q} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t), \mathbf{J} = \tilde{\sigma} \cdot \mathbf{E} \)

\[
\mathbf{n} \times \mathbf{n} \times \mathbf{E} + \left( \frac{\mathbf{I}}{\omega} + \frac{i\tilde{\sigma}}{\epsilon_0\omega} \right) \cdot \mathbf{E} = 0
\]

where \( \mathbf{n} = c\mathbf{k}/\omega \)

By combining this equation with the momentum conservation equations, \( \mathbf{J} = \sum_j n_j q_j v_j \), setting \( p = 0 \), and setting collisionality to 0, it is possible to solve for \( \tilde{\sigma} \). Plugging in \( \tilde{\sigma} \) and then writing in tensor form with Stix notation,

\[
\begin{pmatrix}
S - n^2_{||} & -iD & n_{\perp} n_{||} \\
iiD & S - n^2 & 0 \\
n_{\perp} n_{||} & 0 & P - n^2_{\perp}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
= 0
\]
where \( S = 1 - \sum_j \frac{\omega_{ pj}^2}{\omega^2 - \Omega_j^2} \)

\[ D = \sum_j \frac{\Omega_j}{\omega} \cdot \frac{\omega_{ pj}^2}{\omega^2 - \Omega_j^2} \]

\[ P = 1 - \sum_j \frac{\omega_{ pj}^2}{\omega^2} \]

\[ R = S + D = 1 - \sum_j \frac{\omega_{ pj}^2}{\omega (\omega + \Omega_j)} \]

\[ L = S - D = 1 - \sum_j \frac{\omega_{ pj}^2}{\omega (\omega - \Omega_j)} \]

For a wave propagating at an angle \( \theta \) to \( \hat{b} \)

\[ A n^4 - B n^2 + C = 0 \]

\[ A = S \sin^2 \theta + P \cos^2 \theta \]

\[ B = RL \sin^2 \theta + PS (1 + \cos^2 \theta) \]

\[ C = PRL \]

The equation can be rearranged, conveniently obtaining a handy mnemonic ("P Nar Nal Snarl Nap")

\[ \tan^2(\theta) = -\frac{P (n^2 - R) (n^2 - L)}{(Sn^2 - RL) (n^2 - P)} \]

For a wave traveling at an angle \( \theta \) to \( \hat{b} \), the dispersion relation is given by the Appleton-Hartree equation

\[ n^2 = 1 - \frac{2 \omega_{ pe}^2 \left( \omega^2 - \omega_{ pe}^2 \right)}{2 \omega^2 \left( \omega^2 - \omega_{ pe}^2 \right) - \Omega_e^2 \omega^2 \sin^2 \theta \pm \Omega_e \omega^2 \Sigma} \]

\[ \Sigma = \left[ \Omega_e^2 \sin^4 \theta + 4 \omega^4 \left( 1 - \omega_{ pe}^2 / \omega^2 \right)^2 \cos^2 \theta \right]^{1/2} \]

Polarization of a wave

\[ \frac{iE_x}{E_y} = \frac{n^2 - S}{D} \]

8.1.1 Common Cold Plasma Waves

Defining special frequencies

\[^1\text{Cutoff/Resonances listed for single ion species plasma} \]
8.2. **ELECTROSTATIC WAVES**

<table>
<thead>
<tr>
<th>Name</th>
<th>Dispersion Relation</th>
<th>Type</th>
<th>Resonance</th>
<th>Cutoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light Wave</td>
<td>( n^2 = 1 - \sum_j \frac{\omega^2}{\omega^2_{pj}} )</td>
<td>( \mathbf{B}_0 = 0 )</td>
<td>NA</td>
<td>( \omega_{pe} )</td>
</tr>
<tr>
<td>O wave</td>
<td>( n^2 = P )</td>
<td>( \mathbf{n} \perp \mathbf{B}_0 )</td>
<td>NA</td>
<td>( \omega_{pe} )</td>
</tr>
<tr>
<td>X wave</td>
<td>( n^2 = RL/S )</td>
<td>( \mathbf{n} \perp \mathbf{B}_0 )</td>
<td>( \omega_{uh}, \omega_{lh} )</td>
<td>( \omega_r, \omega_l )</td>
</tr>
<tr>
<td>R wave</td>
<td>( n^2 = R )</td>
<td>( \mathbf{n}</td>
<td></td>
<td>\mathbf{B}_0 )</td>
</tr>
<tr>
<td>L wave</td>
<td>( n^2 = L )</td>
<td>( \mathbf{n}</td>
<td></td>
<td>\mathbf{B}_0 )</td>
</tr>
</tbody>
</table>

\[
\omega_R = \frac{\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}
\]
\[
\omega_L = \frac{-\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}
\]
\[
\omega_{uh}^2 = \omega_p^2 + \Omega_e^2
\]
\[
\omega_{lh}^2 = \Omega_e^2 + \frac{\omega_{pi}^2}{1 + \omega_{pe}/\Omega_e^2}
\]

For waves that are almost one of the O, X, L, R waves, with a small angle with respect to the proper propagation direction, we get the following dispersion relations. In these equations, \( \alpha = \omega_{pe}^2/\omega^2 \).

<table>
<thead>
<tr>
<th>Wave Name</th>
<th>Dispersion Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>QT-O</td>
<td>( n^2 \simeq \frac{1-\alpha}{1-\alpha \cos^2 \theta} )</td>
</tr>
<tr>
<td>QT-X</td>
<td>( n^2 \simeq \frac{(1-\alpha)^2 \omega^2 - \Omega_e^2 \sin^2 \theta}{(1-\alpha)^2 \omega^2 - \Omega_e^2 \sin^2 \theta} )</td>
</tr>
<tr>
<td>QL-L</td>
<td>( n^2 \simeq 1 - \frac{\alpha \omega}{\omega + \Omega_e \cos \theta} )</td>
</tr>
<tr>
<td>QL-R</td>
<td>( n^2 \simeq 1 - \frac{\alpha \omega}{\omega - \Omega_e \cos \theta} )</td>
</tr>
</tbody>
</table>

### 8.2 Electrostatic Waves

These are waves with no perturbed magnetic field. The dispersion relation can be derived by combining Gauss’s law, momentum conservation, and continuity equations. It is important to note that \( \mathbf{k}||\mathbf{E} \). Let \( \mathbf{k} \) lie in the \((x,z)\) plane, where \( z \) points the direction of \( \mathbf{B} \).

\[
1 - \sum_j \left( \frac{\omega_{pj}^2 k_x^2}{\omega^2 - \Omega_j^2 k^2} + \frac{\omega_{pj}^2 k_z^2}{\omega^2 k^2} \right) = 0
\]
Electrostatic wave solutions

<table>
<thead>
<tr>
<th>Solution for $k$</th>
<th>Dispersion relation</th>
<th>Solution</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_x = 0$</td>
<td>$1 - \sum_j \omega_j^2 = 0$</td>
<td>$\omega^2 = \sum_j \omega_j^2$</td>
<td>Plasma oscillations</td>
</tr>
<tr>
<td>$k_z = 0$</td>
<td>$1 - \sum_j \omega_j^2 \xi_j = 0$</td>
<td>$\omega = \omega_{uh}$</td>
<td>Upper hybrid</td>
</tr>
<tr>
<td>$k_x \neq 0$</td>
<td>$\omega^2 = \frac{\omega_{uh}^2}{2} \pm \frac{\omega_{uh}^2}{2} \left( 1 - \frac{4 \Omega^2 \omega_{pe}^2 \cos^2 \theta}{\omega_{uh}^2} \right)^{1/2}$</td>
<td>$\omega = \omega_{lh}$</td>
<td>Lower hybrid</td>
</tr>
<tr>
<td>$k_z \neq 0$</td>
<td>$\omega^2 = \frac{\omega_{lh}^2}{2}$</td>
<td></td>
<td>Trivelpiece-Gould</td>
</tr>
</tbody>
</table>

8.3 MHD Waves

Using the MHD formulation describing plasmas, one can define a perturbation and analyze what waves propagate in a plasma. The matrix equation is given below, where $v_a = B_0/\sqrt{\mu_0 \rho_0}$ and $v_s = \sqrt{\gamma P_0/\rho_0}$. \(^{8:314}\)

$$
\begin{pmatrix}
\omega^2 - k^2 v_a^2 & 0 & 0 \\
0 & \omega^2 - k^2 v_a^2 - k^2 v_s^2 & -k_{\perp} k \| v_s^2 \\
0 & -k_{\perp} k \| v_s^2 & \omega^2 - k^2 v_s^2
\end{pmatrix}
\begin{pmatrix}
\xi_x \\
\xi_y \\
\xi_z
\end{pmatrix}
= 0
$$

MHD wave solutions

<table>
<thead>
<tr>
<th>Solution</th>
<th>Solution ($\beta \ll 1$)</th>
<th>Wave type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^2 = k^2 v_a^2$</td>
<td>$\omega^2 = \frac{k^2}{2} \left( v_a^2 + v_s^2 \right) \left[ 1 \pm (1 - \alpha)^{1/2} \right]$</td>
<td>Shear Alfvén (^{8:314})</td>
</tr>
<tr>
<td>$\omega^2 \approx (k_{\perp}^2 + k^2 v_a^2)$</td>
<td>$\omega^2 \approx k_{\perp}^2 v_a^2$</td>
<td>Compressional (^{8:315})</td>
</tr>
</tbody>
</table>

8.4 Hot Plasma

The hot plasma dispersion relations are calculated using the Vlasov equation and including finite Larmor orbit effects. Hot plasma effects are also characterized by electrostatic and electromagnetic waves.

8.4.1 Electrostatic

The dispersion relation (general plasmas) \(^{19}\)

$$\epsilon(\omega, k) = 1 + \sum_j \chi_j(\omega, k) = 0$$
\[ \chi_j = \frac{\omega_{pj}^2 2 \pi}{k^2 n_{0j}} \int_{-\infty}^{\infty} dv_\parallel \int_{0}^{\infty} v_\perp dv_\perp \sum_{m} J_m^2 \left( k \frac{v_\perp}{\Omega_j} \right) \left[ k_\parallel \frac{\partial f_{0j}}{\partial v_\parallel} + \frac{m \Omega_j \partial f_{0j}}{\partial v_\perp} \right] \]

where the number subscripts refer to the linearization order.

The dispersion relation (Maxwellian plasmas)

\[ \epsilon = 1 + \sum_j \frac{1}{k^2 \lambda_{Dj}^2} \left[ 1 + \zeta_{0j} \sum_{m=-\infty}^{m=\infty} \Gamma_m(b_j) Z(\zeta_{mj}) \right] \]

where

\[ Z(\zeta_{mj}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dv_\parallel e^{v_\parallel^2 / v_{Thj}^2}}{v_\parallel - \left( \omega - m \Omega_j \right) / k_\parallel} \]

\[ b_j = k_\perp^2 \rho_{Lj}^2 \]

\[ \zeta_{mj} = (w - m \Omega_j) / (k_\parallel v_{Thj}) \]

\[ \Gamma_m(b_j) \equiv I_m(b_j) \exp(-b_j) \]

This function can be evaluated as a power or an asymptotic series.

\[ |\zeta_m| \ll 1 \quad \text{(kinetic limit)} \]

\[ \Re Z(\zeta_m) \approx -2\zeta_m \left[ 1 - \frac{2}{3} \zeta_m^2 + \frac{1}{15} \zeta_m^4 + \ldots \right] \quad \text{(power series)} \]

\[ |\zeta_m| \gg 1 \quad \text{(fluid limit)} \]

\[ \Re Z(\zeta_m) \approx -\frac{1}{\zeta_m} \left[ 1 + \frac{1}{2\zeta_m^2} + \frac{3}{4\zeta_m^4} + \ldots \right] \quad \text{(asymptotic)} \]

\[ \Gamma \] can be expanded as well as

\[ \Gamma_0 \approx 1 - b + \frac{3}{4} b^2 + O(b^3) \]

\[ \Gamma_1 \approx \frac{b}{2} (1 - b) + O(b^3) \]

\[ \Gamma_2 \approx b^2 / 8 + O(b^3) \]

\[ \Gamma_3 \approx O(b^3) \]
8.4.2 Electromagnetic

The dispersion relation in the electromagnetic limit

\[
\begin{pmatrix}
K_{xx} - n_z^2 & K_{xy} \\
K_{yz} & K_{yy} - n_z^2 - n_x^2 \\
K_{zx} + n_x n_z & K_{zy}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix} = 0
\]

\[
K_{xx} = 1 + \sum_{j=i,e} \sum_{k=-\infty}^{\infty} \frac{l^2 \Omega^3}{k^2_{\perp}} \int dv^2 J_l^2(\lambda) P_l
\]

\[
K_{xy} = -K_{yx} = i \sum_j \sum_{l=\infty}^{\infty} \frac{l \Omega^2}{k^2_{\perp}} \int dv^2 v_\perp J_l(\lambda) J_l'(\lambda) P_l
\]

\[
K_{xz} = \sum_j \sum_{l=-\infty}^{\infty} \frac{l \Omega^2}{k^2_{\perp}} \int dv^2 v_\parallel J_l^2(\lambda) Q_l
\]

\[
K_{yy} = 1 + \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 \left[ v_\perp J_l(\lambda) \right]^2 P_l
\]

\[
K_{yz} = -i \sum_j \sum_{l=\infty}^{\infty} \Omega \int dv^2 v_\perp v_\parallel J_l(\lambda) J_l'(\lambda) Q_l
\]

\[
K_{zx} = \sum_j \sum_{l=-\infty}^{\infty} \frac{l \Omega^2}{k^2_{\perp}} \int dv^2 v_\parallel J_l^2(\lambda) P_l
\]

\[
K_{zy} = i \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 v_\perp v_\parallel J_l(\lambda) J_l'(\lambda) P_l
\]

\[
K_{zz} = 1 + \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 v_\parallel^2 J_l^2(\lambda) Q_l
\]

\[
\int dv^2 \equiv 2 \int_{-\infty}^{\infty} dv_\parallel \int_{0}^{\infty} v_\perp dv_\perp
\]

\[
P_l = \frac{2\pi}{\omega \Omega_j} \frac{\omega_p^2}{\omega_j} \left[ \frac{\partial f_0}{\partial v_\perp} + \frac{k_{\perp} v_\perp}{\omega} \left( \frac{\partial f_0}{\partial v_\parallel} - \frac{\partial f_0}{\partial v_\perp} \right) \right]
\]

\[
Q_l = \frac{2\pi}{\omega \Omega_j} \frac{\omega_p^2}{\omega_j} \left[ \frac{\partial f_0}{\partial v_\parallel^2} - \frac{\Omega_j}{\omega} \left( \frac{\partial f_0}{\partial v_\parallel} - \frac{\partial f_0}{\partial v_\perp} \right) \right]
\]

where \( \lambda = |k_{\perp} v_\perp/\Omega| \) and \( \partial f/\partial v^2 \equiv \partial f/\partial (v^2) \)
Dispersion relation evaluation for an isotropic Maxwellian plasma

\[ K_{xx} = 1 + \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-bj}}{b_j} \zeta_0 \sum_{n=-\infty}^{\infty} n^2 I_n(b_j) Z(\zeta_n) \]

\[ K_{xy} = -K_{yz} = i \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-bj}}{b_j} \zeta_0 \sum_{n=-\infty}^{\infty} n \left[ I_n(b_j) - I'_n(b_j) \right] Z(\zeta_n) \]

\[ K_{xz} = K_{zx} = \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-bj}}{b_j (2b_j)^{1/2}} \zeta_0 \sum_{n=-\infty}^{\infty} n I_n(b_j) Z'(\zeta_n) \]

\[ K_{yy} = 1 + \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-bj}}{b_j} \zeta_0 \sum_{n=-\infty}^{\infty} \left( n^2 I_n(b_j) + 2b_j^2 \left[ I_n(b_j) - I'_n(b_j) \right] \right) Z(\zeta_n) \]

\[ K_{yz} = -K_{zy} = -i \sum_j \frac{\omega_{pj}^2}{\omega^2} \left( \frac{b_j}{2} \right)^{1/2} e^{-bj} \zeta_0 \sum_{n=-\infty}^{\infty} \left[ I_n(b_j) - I'_n(b_j) \right] Z'(\zeta_n) \]

\[ K_{zz} = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-bj}}{\zeta_0} \sum_{n=-\infty}^{\infty} I_n(b_j) \zeta_n Z'(\zeta_n) \]
Chapter 9

Nuclear Physics

In this chapter, all units are SI with the exception of: temperature and energy, which are defined in the historical units of eV (electron-volts); cross sections, which are defined in the historical units of barn; and Bosch-Hale reaction rates, which are given in cubic centimeters per second.

\( e \) is the elementary electric charge
\( Z \) is the number of nuclear protons
\( N \) is the number of nuclear neutrons
\( A \) is the number of nucleons \((N+Z)\)
\( m \) is the particle mass
\( n \) is the particle number density
\( \mathbf{v} \) is the particle velocity vector
\( E \) is the particle kinetic energy

9.1 Fundamental Definitions

Nuclear reaction notation \(^{15}:378-381\)

\[
\begin{align*}
\text{a} + X & \Rightarrow b + Y \\
\text{or} & \\
X(a,b)Y & \text{where} \\
& \begin{cases} 
  a = \text{bombarding particle} \\
  X = \text{target nucleus} \\
  b = \text{ejected particle(s)} \\
  Y = \text{product nucleus}
\end{cases}
\end{align*}
\]

\( X(a,b)Y \) general nuclear reaction
\( X(a,a)X \) elastic scattering
\( X(a,a')X^* \) inelastic scattering
\( X(n,n)X \) neutron elastic scattering
\( X(n,n')X^* \) neutron inelastic scattering
\( X(n,2n)Y \) neutron multiplication
\( X(n,\gamma)Y \) neutron capture
\( X(n,f)Y \) neutron-induced fission
Nuclear mass

\[ m = Zm_p + Nm_n - E_B/c^2 \]

Nuclear binding energy

\[ E_B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z) \]

where the value of the coefficients in MeV is

\[ a_v = 15.5 \]
\[ a_s = 16.8 \]
\[ a_c = 0.72 \]
\[ a_a = 23 \]
\[ a_p = 34 \text{ MeV} \]

\[ \delta(A, Z) = \begin{cases} 
    a_p A^{-3/4} & \text{(Z,N even)} \\
    0 & \text{(A odd)} \\
    -a_p A^{-3/4} & \text{(Z,N odd)}
\end{cases} \]

Nuclear reaction Q-value

\[ Q = ((m_a + m_X) - (m_b + m_Y))c^2 = E_f - E_i \]

Reaction threshold energy \((Q < 0)\)

\[ E_{\text{thresh}} = -\left( \frac{m_Y + m_b}{m_Y + m_b - m_a} \right) Q \approx -\left( 1 + \frac{m_a}{m_X} \right) Q \]

9.2 Nuclear Interactions

\( A \equiv m_A/m_N \) is approximately the nucleus atomic mass number

\( i \) and \( f \) refer to initial and final, respectively

\( \theta \) is the angle between the \( a \) and \( b \) trajectory in the lab frame

\( \theta_{\text{cm}} \) is the angle between the \( a \) and \( b \) trajectory in the center of mass frame

Reaction Q-value from kinematics

\[ Q = E_b \left( 1 + \frac{m_b}{m_Y} \right) - E_a \left( 1 - \frac{m_a}{m_Y} \right) - 2 \left( \frac{m_a m_b}{m_Y^2} E_a E_b \right)^{1/2} \cos \theta \]

Ejected particle energy: \( X(a,b)Y \)

\[ E_y^{1/2} = \xi \left( \frac{m_a m_b E_a}{m_Y + m_b} \right)^{1/2} \pm \frac{(m_a m_b E_a \xi^2 + (m_Y + m_b)[m_Y Q + (m_Y - m_a)E_a])^{1/2}}{m_Y + m_b} \]

Maximum kinetic energy transfer fraction: \( X(a,a)X \)

\[ \left. \frac{E_f}{E_i} \right|_{\theta=0} = \frac{4A_1 A_2}{(A_1 + A_2)^2} \]
9.2. NUCLEAR INTERACTIONS

9.2.1 Charged Particle Interactions

Stopping power of heavy charged particles (Bethe formula)\textsuperscript{15:194}

\[
\frac{dE}{dx} = \left( \frac{e^2}{4\pi\varepsilon_0} \right)^2 4\pi z^2 N_A Z \rho \frac{2m_e c^2 \beta^2}{I} \left[ \ln \left( \frac{2m_e c^2 \beta^2}{I} \right) - \ln \left( 1 - \beta^2 \right) - \beta^2 \right]
\]

where $\beta c = v$ and $ze$ are the incident particle’s speed and charge; $\rho$ is the mass density of the stopping medium; $N_A$ is Avogadro’s number; $I$ is mean excitation of atomic electrons, typically taken as $I \approx 10Z$.

Collisional stopping power of electrons\textsuperscript{15:196}

\[
\left( \frac{dE}{dx} \right)_c = \left( \frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{2\pi N_A Z \rho}{m_e c^2 \beta^2 A} \left[ \ln \frac{E_i (E_i + m_e c^2)^2 \beta^2}{2I^2 m_e c^2} + (1 - \beta^2) \right. \\
\left. - \left( 2\sqrt{1 - \beta^2} - 1 + \beta^2 \right) \ln 2 + \frac{1}{8} \left( 1 - \sqrt{1 - \beta^2} \right)^2 \right]
\]

Radiative stopping power of electrons ($\gtrsim 1$ MeV)\textsuperscript{15:196}

\[
\left( \frac{dE}{dx} \right)_r = \left( \frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{Z^2 N_A (E_i + m_e c^2) \rho}{137m_e^2 c^4 A} \left[ 4 \ln \frac{2(E_i + m_e c^2)}{m_e c^2} - \frac{4}{3} \right]
\]

9.2.2 Neutron Interactions

Ejected neutron energy: X(n,n)X\textsuperscript{15:448}

\[
E_{n,f} = \frac{A^2 + 2A \cos \theta_{em} + 1}{(A + 1)^2}
\]

Maximum kinetic energy transfer fraction: X(n,n)X\textsuperscript{15:448}

\[
\left. \frac{E_{n,f}}{E_{n,i}} \right|_{\theta=0} = \left( \frac{A - 1}{A + 1} \right)^2
\]

Neutron lethargy ($E_n \lesssim 10$ MeV)\textsuperscript{15:450}

\[
\xi = 1 + \frac{(A - 1)^2}{2A} \ln \left( \frac{A - 1}{A + 1} \right)
\]

Number of collisions required to change neutron energies\textsuperscript{15:450}

\[
N = \frac{\ln(E_i / E_f)}{\xi}
\]
9.2.3 Gamma Interactions

Klein-Nishina cross section for Compton scattering 15:201

$$\frac{d\sigma}{d\Omega} = r_0^2 \left[ \frac{1}{1 + \alpha (1 - \cos \theta)} \right]^3 \left[ \frac{1 + \cos^2 \theta}{2} \right] \left[ 1 + \frac{\alpha^2 (1 - \cos \theta)}{(1 + \cos^2 \theta) [1 + \alpha (1 - \cos \theta)]} \right]$$

where ($\alpha = E_\gamma/mc^2$) and ($r_0 = e^2/4\pi\epsilon_0 mc^2 = 2.818$ fm) is the classical electron radius.

Energy shift from Compton scattering 15:201

$$E_f = \frac{E_i}{1 + \left( \frac{E_i}{mc^2} \right) (1 - \cos \theta)}$$

9.3 Cross Section Theory

A reaction cross section $\sigma(E_a)$, or more simply $\sigma$, is a measure of the probability that reaction $X(a,b)Y$ will occur. Consider a beam of $a$ particle with current $I_a$ directed onto $X$ target nuclei with an areal density of $N_X$ per unit area. By observing the energy and angular distribution, $e(E_b)$ and $r(\theta, \phi)$ respectively, of the ejected particle $b$ into a solid angle $d\Omega$ then the doubly differential cross section can be determined, which is the probability of observing particle $b$ at in solid angle $d\Omega$ with energy $E_b$. 15:392–394

Doubly differential cross section 15:392–393

$$\frac{d\sigma}{d\Omega dE_b} = \frac{r(\theta, \phi) e(E_b)}{4\pi I_a N_X}$$

Differential energy cross section 15:393

$$\frac{d\sigma}{dE_b} = \int \frac{d\sigma}{d\Omega dE_b} d\Omega$$

Differential angular cross section 15:393

$$\frac{d\sigma}{d\Omega} = \int \frac{d\sigma}{d\Omega dE_b} dE_b$$

Reaction cross section 15:393

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int \int \frac{d\sigma}{d\Omega} \sin \theta d\theta d\phi$$

Total cross section 15:393

$$\sigma_T = \sum_{i=0} \sigma_i$$
The Sommerfeld parameter
\[ \eta = \frac{Z_a Z_X e^2}{\hbar v} \]

Astrophysical S-factor
\[ S(E) = \sigma(E) E \exp(2\pi \eta) \]

9.4 Reaction Rate Theory

For a thermonuclear plasma, the volumetric reaction rate \( R \) (also known as thermal reactivity) describes the number of reactions occurring per unit volume per unit time. In thermonuclear fusion plasmas, \( R \) is obtained by integrating the energy-dependent cross section, \( \sigma(v) \), over the distribution functions of the participating species.

Partial volumetric reaction rate
\[ R_p = \sigma(v) v f_1(v_1) f_2(v_2) \quad v = v_1 - v_2 \]

Total volumetric reaction rate for general \( f(v) \)
\[ R = \frac{1}{1 + \delta_{12}} \int \sigma(v) v f_1(v_1) f_2(v_2) d^3v_1 d^3v_2 \]

where \( \delta_{12} \) is the Kronecker delta for particle species 1 and 2 that prevents double counting for like-species plasmas.

Total volumetric reaction rate for Maxwellian \( f(v) \)
\[ R = \frac{n_1 n_2}{1 + \delta_{12}} \langle \sigma v \rangle \text{ where } \left\{ \begin{array}{l} \langle \sigma v \rangle = \left( \frac{8\mu^3}{\pi T^3} \right)^{1/2} \frac{1}{m_1^3} \int \sigma(E) E e^{-\mu E/m_1 T} dE \quad \text{[m^3s^{-1}]} \\ \mu = \frac{m_1 m_2}{m_1 + m_2} \\ E = \frac{1}{2} m v^2 \end{array} \right. \]
9.5 Nuclear Reactions for Fusion Plasmas

All data from the ENDF/B-VII nuclear data libraries.  

<table>
<thead>
<tr>
<th>Reactants</th>
<th>Products (kinetic energy in MeV)</th>
<th>Branching ratio</th>
<th>Q-value (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>d + t</td>
<td>$\alpha$(3.52) + n(14.07)</td>
<td>1.00</td>
<td>17.59</td>
</tr>
<tr>
<td>d + d</td>
<td>t(1.01) + p(3.02)</td>
<td>0.50</td>
<td>4.03</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow$ h(0.82) + n(2.45)</td>
<td>0.50</td>
<td>3.27</td>
</tr>
<tr>
<td>d + h</td>
<td>$\alpha$(3.67) + p(14.68)</td>
<td>1.00</td>
<td>18.35</td>
</tr>
<tr>
<td>t + t</td>
<td>$\alpha$ + 2n</td>
<td>1.00</td>
<td>11.33</td>
</tr>
<tr>
<td>h + t</td>
<td>$\alpha$ + p + n</td>
<td>0.51</td>
<td>12.10</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow$ $\alpha$(4.77) + d(9.54)</td>
<td>0.43</td>
<td>14.32</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow$ $^5$He(1.87) + p(9.34)</td>
<td>0.06</td>
<td>11.21</td>
</tr>
<tr>
<td>p + $^6$Li</td>
<td>$\alpha$(1.72) + h(2.30)</td>
<td>1.00</td>
<td>4.02</td>
</tr>
<tr>
<td>p + $^7$Li</td>
<td>$\rightarrow$ 2 $\alpha$</td>
<td>0.20</td>
<td>17.35</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow$ $^7$Be + n</td>
<td>0.80</td>
<td>-1.64</td>
</tr>
<tr>
<td>d + $^6$Li</td>
<td>$\rightarrow$ 2 $\alpha$</td>
<td>1.00</td>
<td>22.37</td>
</tr>
<tr>
<td>p + $^{11}$B</td>
<td>$\rightarrow$ 3 $\alpha$</td>
<td>1.00</td>
<td>8.62</td>
</tr>
</tbody>
</table>

9.6 Nuclear Reactions for Fusion Energy

All data from the ENDF/B-VII nuclear data libraries.  

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Q-value [MeV]</th>
<th>Purpose</th>
<th>$\sigma$(0.025 eV) [barn]</th>
<th>$\sigma$(14.1 MeV) [barn]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^6$Li(n,t)$^4$He</td>
<td>4.78</td>
<td>B</td>
<td>978</td>
<td>0.03</td>
</tr>
<tr>
<td>$^6$Li(n,2n)$^6$Li</td>
<td>-3.96</td>
<td>M</td>
<td>-</td>
<td>0.08</td>
</tr>
<tr>
<td>$^7$Li(n,2n)$^6$Li</td>
<td>-7.25</td>
<td>M</td>
<td>-</td>
<td>0.03</td>
</tr>
<tr>
<td>$^7$Li(n,2n)$^3$H</td>
<td>-8.72</td>
<td>B/M</td>
<td>-</td>
<td>0.02</td>
</tr>
<tr>
<td>$^9$Be(n,2n)$^9$Be</td>
<td>-1.57</td>
<td>M</td>
<td>-</td>
<td>0.48</td>
</tr>
<tr>
<td>$^{204}$Pb(n,2n)$^{203}$Pb</td>
<td>-8.39</td>
<td>M</td>
<td>-</td>
<td>2.22</td>
</tr>
<tr>
<td>$^{206}$Pb(n,2n)$^{205}$Pb</td>
<td>-8.09</td>
<td>M</td>
<td>-</td>
<td>2.22</td>
</tr>
<tr>
<td>$^{207}$Pb(n,2n)$^{206}$Pb</td>
<td>-6.74</td>
<td>M</td>
<td>-</td>
<td>2.29</td>
</tr>
<tr>
<td>$^{208}$Pb(n,2n)$^{207}$Pb</td>
<td>-7.37</td>
<td>M</td>
<td>-</td>
<td>2.30</td>
</tr>
</tbody>
</table>
9.7 Fusion Cross Section Parametrization

The Bosch-Hale parametrization of the fusion reaction cross section

\[ \sigma(E_{\text{keV}}) = \frac{S(E)}{E \exp(B_G/\sqrt{E})} \quad \text{[millibarn]} \]

where

\[ S(E_{\text{keV}}) = \frac{A_1 + E(A_2 + E(A_3 + E(A_4 + E(A_5))))}{1 + E(B_1 + E(B_2 + E(B_3 + EB_4)))} \]

Bosch-Hale parametrization coefficients for several fusion reactions

<table>
<thead>
<tr>
<th>( B_G ) [keV]</th>
<th>( ^2\text{H}(d,n)^3\text{He} )</th>
<th>( ^2\text{H}(d,p)^3\text{H} )</th>
<th>( ^3\text{H}(d,n)^4\text{He} )</th>
<th>( ^3\text{He}(d,p)^4\text{He} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.397</td>
<td>5.3701×10^4</td>
<td>5.5576×10^4</td>
<td>6.927×10^4</td>
<td>5.7501×10^6</td>
</tr>
<tr>
<td>31.397</td>
<td>3.1054×10^2</td>
<td>7.454×10^6</td>
<td>2.5226×10^3</td>
<td></td>
</tr>
<tr>
<td>34.3827</td>
<td>-2.638×10^-2</td>
<td>2.050×10^6</td>
<td>4.5666×10^1</td>
<td></td>
</tr>
<tr>
<td>68.7508</td>
<td>1.4987×10^-5</td>
<td>5.200×10^3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>-2.5151×10^-9</td>
<td>1.8181×10^-10</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

Valid Range [keV] 0.5\(<E<90000\) 0.5\(<E<50000\) 0.5\(<E<5500\) 0.3\(<E<900\)

Tabulated Bosch-Hale cross sections [millibarns]

<table>
<thead>
<tr>
<th>E (keV)</th>
<th>( ^2\text{H}(d,n)^3\text{He} )</th>
<th>( ^2\text{H}(d,p)^3\text{H} )</th>
<th>( ^3\text{H}(d,n)^4\text{He} )</th>
<th>( ^3\text{He}(d,p)^4\text{He} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.445×10^-4</td>
<td>2.513×10^-4</td>
<td>9.808×10^-3</td>
<td>1.119×10^-11</td>
</tr>
<tr>
<td>4</td>
<td>2.093×10^-3</td>
<td>2.146×10^-3</td>
<td>1.073×10^-1</td>
<td>1.718×10^-9</td>
</tr>
<tr>
<td>5</td>
<td>8.834×10^-3</td>
<td>9.038×10^-3</td>
<td>5.383×10^-1</td>
<td>5.199×10^-8</td>
</tr>
<tr>
<td>6</td>
<td>2.517×10^-2</td>
<td>2.569×10^-2</td>
<td>1.749×10^0</td>
<td>6.336×10^-7</td>
</tr>
<tr>
<td>7</td>
<td>5.616×10^-2</td>
<td>5.720×10^-2</td>
<td>4.335×10^0</td>
<td>4.373×10^-6</td>
</tr>
<tr>
<td>8</td>
<td>1.064×10^-1</td>
<td>1.081×10^-1</td>
<td>8.968×10^0</td>
<td>2.058×10^-5</td>
</tr>
<tr>
<td>9</td>
<td>1.794×10^-1</td>
<td>1.820×10^-1</td>
<td>1.632×10^1</td>
<td>7.374×10^-5</td>
</tr>
<tr>
<td>10</td>
<td>2.779×10^-1</td>
<td>2.812×10^-1</td>
<td>2.702×10^1</td>
<td>2.160×10^-4</td>
</tr>
<tr>
<td>12</td>
<td>5.563×10^-1</td>
<td>5.607×10^-1</td>
<td>6.065×10^2</td>
<td>1.206×10^-3</td>
</tr>
<tr>
<td>15</td>
<td>1.178×10^0</td>
<td>1.180×10^0</td>
<td>1.479×10^2</td>
<td>7.944×10^-3</td>
</tr>
<tr>
<td>20</td>
<td>2.691×10^0</td>
<td>2.670×10^0</td>
<td>4.077×10^2</td>
<td>6.568×10^-2</td>
</tr>
</tbody>
</table>
9.8 Fusion Reaction Rate Parametrization

The Bosch-Hale parametrization of the volumetric reaction rates

\[
\langle \sigma v \rangle = C_1 \cdot \theta \cdot \sqrt{\frac{\xi}{m_\nu c^2 T_i \text{ keV}}} e^{-3\xi} \quad [\text{cm}^3 \text{ s}^{-1}] \quad \text{or} \quad \times 10^{-6} [\text{m}^3 \text{ s}^{-1}]
\]

where

\[
\theta = T/\left(1 - \frac{T(C_2 + T(C_4 + TC_6))}{1 + T(C_3 + T(C_5 + TC_7))}\right) \quad \xi = \left(\frac{B_G^2}{4\theta}\right)^{1/3}
\]

Bosch-Hale parametrization coefficients for volumetric reaction rates

<table>
<thead>
<tr>
<th>(B_G) [keV^{1/2}]</th>
<th>(^2\text{H}(\text{d,n})^3\text{He})</th>
<th>(^2\text{H}(\text{d,p})^3\text{H})</th>
<th>(^3\text{H}(\text{d,n})^4\text{He})</th>
<th>(^3\text{He}(\text{d,p})^4\text{He})</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.3970</td>
<td>31.3970</td>
<td>34.3827</td>
<td>68.7508</td>
<td></td>
</tr>
<tr>
<td>937 814</td>
<td>937 814</td>
<td>1 124 656</td>
<td>1 124 572</td>
<td></td>
</tr>
<tr>
<td>(C_1)</td>
<td>5.43360\times10^{-12}</td>
<td>5.65718\times10^{-12}</td>
<td>1.17302\times10^{-9}</td>
<td>5.51036\times10^{-10}</td>
</tr>
<tr>
<td>(C_2)</td>
<td>5.85778\times10^{-3}</td>
<td>3.41267\times10^{-3}</td>
<td>1.51361\times10^{-2}</td>
<td>6.41918\times10^{-3}</td>
</tr>
<tr>
<td>(C_3)</td>
<td>7.68222\times10^{-3}</td>
<td>1.99167\times10^{-3}</td>
<td>7.51886\times10^{-2}</td>
<td>-2.02896\times10^{-3}</td>
</tr>
<tr>
<td>(C_4)</td>
<td>0.0</td>
<td>0.0</td>
<td>4.60643\times10^{-3}</td>
<td>-1.91080\times10^{-5}</td>
</tr>
<tr>
<td>(C_5)</td>
<td>-2.96400\times10^{-6}</td>
<td>1.05060\times10^{-5}</td>
<td>1.35000\times10^{-2}</td>
<td>1.35776\times10^{-4}</td>
</tr>
<tr>
<td>(C_6)</td>
<td>0.0</td>
<td>0.0</td>
<td>-1.06750\times10^{-4}</td>
<td>0.0</td>
</tr>
<tr>
<td>(C_7)</td>
<td>0.0</td>
<td>0.0</td>
<td>1.36600\times10^{-5}</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Valid range (keV) \(0.2<T_i<100\) \(0.2<T_i<100\) \(0.2<T_i<100\) \(0.5<T_i<190\)

Tabulated Bosch-Hale reaction rates [m\(^3\) s\(^{-1}\)]

<table>
<thead>
<tr>
<th>(T) (keV)</th>
<th>(^2\text{H}(\text{d,n})^3\text{He})</th>
<th>(^2\text{H}(\text{d,p})^3\text{H})</th>
<th>(^3\text{H}(\text{d,n})^4\text{He})</th>
<th>(^3\text{He}(\text{d,p})^4\text{He})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>9.933\times10^{-29}</td>
<td>1.017\times10^{-28}</td>
<td>6.857\times10^{-27}</td>
<td>3.057\times10^{-32}</td>
</tr>
<tr>
<td>1.5</td>
<td>8.284\times10^{-28}</td>
<td>8.431\times10^{-28}</td>
<td>6.923\times10^{-26}</td>
<td>1.317\times10^{-30}</td>
</tr>
<tr>
<td>2.0</td>
<td>3.110\times10^{-27}</td>
<td>3.150\times10^{-27}</td>
<td>2.977\times10^{-25}</td>
<td>1.399\times10^{-29}</td>
</tr>
<tr>
<td>3.0</td>
<td>1.602\times10^{-26}</td>
<td>1.608\times10^{-26}</td>
<td>1.867\times10^{-24}</td>
<td>2.676\times10^{-26}</td>
</tr>
<tr>
<td>4.0</td>
<td>4.447\times10^{-26}</td>
<td>4.428\times10^{-26}</td>
<td>5.974\times10^{-24}</td>
<td>1.710\times10^{-27}</td>
</tr>
<tr>
<td>5.0</td>
<td>9.128\times10^{-26}</td>
<td>9.024\times10^{-26}</td>
<td>1.366\times10^{-23}</td>
<td>6.377\times10^{-27}</td>
</tr>
<tr>
<td>6.0</td>
<td>3.457\times10^{-25}</td>
<td>3.354\times10^{-25}</td>
<td>6.222\times10^{-23}</td>
<td>7.504\times10^{-26}</td>
</tr>
<tr>
<td>10.0</td>
<td>6.023\times10^{-25}</td>
<td>5.781\times10^{-25}</td>
<td>1.136\times10^{-22}</td>
<td>2.126\times10^{-25}</td>
</tr>
<tr>
<td>12.0</td>
<td>9.175\times10^{-25}</td>
<td>8.723\times10^{-25}</td>
<td>1.747\times10^{-22}</td>
<td>4.715\times10^{-25}</td>
</tr>
<tr>
<td>15.0</td>
<td>1.481\times10^{-24}</td>
<td>1.390\times10^{-24}</td>
<td>2.740\times10^{-22}</td>
<td>1.175\times10^{-24}</td>
</tr>
<tr>
<td>20.0</td>
<td>2.603\times10^{-24}</td>
<td>2.399\times10^{-24}</td>
<td>4.330\times10^{-22}</td>
<td>3.482\times10^{-24}</td>
</tr>
</tbody>
</table>

Approximate DT volumetric reaction rate (10 \(\lesssim T\) [keV] \(\lesssim 20\))

\[
\langle \sigma v \rangle_{\text{DT}} = 1.1 \times 10^{-24} T_{\text{keV}}^2 \quad [\text{m}^3 \text{ s}^{-1}]
\]
9.9 Cross Section and Reaction Rate Plots

9.9.1 Bosch-Hale Nuclear Reaction Cross Sections

9.9.2 Bosch-Hale Volumetric Reaction Rates
Chapter 10

Tokamak Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the fundamental charge unit
Z is the number of nuclear protons
R_0 is the major radius of a toroidal plasma
a and b are the horizontal and vertical minor radii of a toroidal plasma

Note: a = b for circular cross sections, and a is used by convention
R and r denote lengths in the major and minor radii, respectively
θ and φ are the poloidal and toroidal angular coordinates, respectively
B_x is the magnetic field in direction x
B_{xa} is the magnetic field evaluated at the plasma edge in direction x
I_p is the toroidal plasma current
p is the plasma pressure
v is velocity of the plasma

10.1 Fundamental Definitions

Inverse aspect ratio \(^{26}:117\)
\[ \epsilon = \frac{a}{R_0} \]

Plasma elongation \(^{26}:741\)
\[ \kappa = \frac{b}{a} \]

Plasma triangularity \(^{26}:741\)
\[ \delta = \frac{(c + d)/2}{a} \]

where c, d are distances to the top of the plasma and the x-point, respectively, from the plasma center.
Large aspect ratio expansion ($\epsilon \ll 1$)  
\[
\frac{1}{R} \approx \frac{1}{R_0} \left( 1 - \frac{r}{R_0} \cos \theta \right)
\]

Surface area of a torus

\[
S_{c\text{-torus}} = 4\pi^2 aR_0 \quad \text{(Circular cross section)}
\]
\[
S_{e\text{-torus}} = 8\pi aR_0 E(k) \approx 4\pi^2 aR_0 \left( \frac{1 + \kappa^2}{2} \right)^{1/2} \quad \text{(Elliptical cross section)}
\]

Volume of a torus

\[
V_{c\text{-torus}} = 2\pi^2 a^2 R_0 \quad \text{(Circular cross section)}
\]
\[
V_{e\text{-torus}} = 2\pi^2 a^2 \kappa R_0 \quad \text{(Elliptical cross section)}
\]

MHD toroidal plasma volume

\[
V(\psi) = \frac{2\pi^2 \dot{r}}{0} \int_0^{2\pi} \int_0^{2\pi} Rrd\theta d\phi
\]
\[
= \pi R_0^{2\pi} \int_0^{2\pi} d\theta R^2 \left[ 1 + \frac{2}{3} \left( \frac{\dot{r}}{R_0} \right) \cos \theta \right] \quad \text{(low aspect ratio)}
\]

Volume averaged plasma pressure

\[
\langle p \rangle = \frac{1}{V} \int_{\text{volume}} p \, d\tau
\]

Toroidal plasma beta

\[
\beta_t = \frac{2\mu_0 \langle p \rangle}{B_{\phi a}^2}
\]

Poloidal plasma beta

\[
\beta_p = \frac{2\mu_0 \langle p \rangle}{B_{\phi a}^2} = \frac{8\pi^2 a^2 \kappa^2 \langle p \rangle}{\mu_0 I_p^2}
\]

Radial electric field in a rotating toroidal plasma

\[
E_r \approx v_\phi B_\theta - v_\theta B_\phi + \frac{1}{Z_i e n} \nabla p
\]
10.2 Magnetic Topology

Toroidal magnetic field for plasma confinement

\[ B_\phi \approx \frac{B_\phi(r)R_0}{R} \]
\[ \approx \frac{B_\phi(r)}{1 - \epsilon \cos \theta} \quad \text{(valid for } \epsilon \ll 1) \]

Poloidal magnetic field for plasma confinement

\[ B_\theta \approx \frac{\mu_0 I_p(r)}{2\pi r} \]

Safety factor (general)

\[ q(r) = \frac{\# \text{ of toroidal field line orbits at } r}{\# \text{ of poloidal field line orbits at } r} \]

Safety factor for cylindrical plasma \((r, \theta, z)\)

\[ q(r)_{cyl} = \frac{r B_\phi(r)}{RB_\theta(r)} = \frac{2\pi r^2 B_\phi(r)}{\mu_0 I_p(r)R} \]

Safety factor for toroidal plasma \((R, \theta, \phi)\)

\[ q(r^*)_{tor} = \frac{1}{2\pi} \int_0^{2\pi} \frac{r B_\phi}{RB_\theta} \ d\theta \]
\[ = \frac{r_0 B_\phi(r_0)}{R_0 B_\theta(r_0) \left(1 - r_0^2/R_0^2\right)^{1/2}} \]

where the flux surfaces \(r^* = r_0\) are circles.

Safety factor at the edge for toroidal plasma \((R, \theta, \phi)\)

\[ q_* = \frac{2\pi a^2 B_0}{\mu_0 R_0 I_p} = \frac{\pi k}{4E(k)(\beta/\epsilon)^{1/2}} \]

where \(E(k) \approx \left[k^2 + \pi^2/4(1 - k^2)\right]\) is the complete elliptic integral of the second kind and the definition of \(k\) is given by

\[ \frac{B^2}{B_0^2} = 1 - \frac{2\mu_0 p}{B_0^2} + \frac{4\epsilon \mu_0 p}{k^2 B_0^2} (2 - k^2) \]

Approximate edge safety factor for a large aspect ratio toroidal plasma

\[ q_* \approx \frac{2\pi B_0 a^2}{\mu_0 R_0 I_p} \left(\frac{1 + \kappa^2}{2}\right) \]

Magnetic shear

\[ s = \frac{r \ dq}{q \ dr} \]
10.3 Magnetic Inductance

Definition of magnetic inductance:  
\[ \frac{1}{2} L I^2 \equiv \int_{\text{volume}} \frac{B^2}{2\mu_0} d\tau \]

Normalized inductance per unit length [dimensionless]:  
\[ \ell = \frac{L}{2\pi R_0} \frac{2L}{\mu_0 R_0} \]

Internal inductance of a toroidal plasma:  
\[ L_i = \frac{8\pi R_0}{I_p^2} \int_0^a B_\theta^2 r dr = \frac{\mu_0 R_0 \langle B_\theta^2 \rangle}{2B_\theta^2 a} \]
\[ \ell_i = \frac{\langle B_\theta^2 \rangle}{B_\theta^2(a)} \]

External inductance of a toroidal plasma:  
\[ L_e = \frac{8\pi R_0}{I_p^2} \int_a^\infty B_\theta^2 r dr = \mu_0 R_0 \left( \ln \frac{8R_0}{a} - 2 \right) \]
\[ \ell_e = 2 \ln \frac{8R_0}{a} - 4 \]

10.4 Toroidal Force Balance

Equation of toroidal force balance:  
\[ \int \hat{R} \cdot (J \times \mathbf{B} - \nabla p) d\tau = 0 \]
where  
\[ \mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \frac{R_0}{R} \frac{\partial B_\phi}{\partial r} \hat{\phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{R_0}{R} r B_\theta \right) \hat{\phi} \]
\[ \hat{R} \cdot J \times B = -\cos \theta \left[ \frac{R_0^2}{R} \frac{\partial}{\partial r} \left( \frac{B_\phi^2}{2\mu_0} \right) + \frac{R_0 B_\theta}{\mu_0 r R} \frac{\partial}{\partial r} \left( \frac{R_0}{R} r B_\theta \right) \right] - \frac{B_{\text{vert}}}{\mu_0 r} \frac{\partial}{\partial r} \left( \frac{R_0}{R} r B_\theta \right) \]

The hoop force:  
\[ \mathbf{F}_{\text{hoop}} = \frac{I_p^2}{2} \frac{\partial}{\partial R} (L_i + L_e) \hat{R} = 2\pi^2 a^2 (\ell_i + \ell_e + 2) \frac{B_{\theta a}}{2\mu_0} \hat{R} \]
\[ = \frac{\mu_0 I_p^2}{2} \left( \ln \frac{8R_0}{a} - 1 + \frac{\ell_i}{2} \right) \hat{R} \]
10.5. PLASMA P ARA- AND DIA-MAGNETISM

The tire tube force

$$F_{\text{tire}} = 2\pi^2 a^2 \langle p \rangle \hat{\mathbf{R}}$$

$$= \frac{\mu_0 I_p^2}{4} \beta_p \hat{\mathbf{R}}$$

The 1/R force

$$F_{1/R} = 2\pi^2 a^2 \left( \frac{B_{\phi a}^2}{2\mu_0} - \frac{\langle B_{\phi}^2 \rangle}{2\mu_0} \right) \hat{\mathbf{R}}$$

$$= \frac{\mu_0 I_p^2}{4} (\beta_p - 1) \hat{\mathbf{R}}$$

where $$\langle p \rangle = \frac{1}{2\mu_0} \left( B_{\phi a}^2 - \langle B_{\phi}^2 \rangle + B_{\theta a}^2 \right)$$ have been used.

The vertical field force on toroidal plasma ring

$$F_{\text{vert}} = -2\pi R_0 B_{\text{vert}} I_p \hat{\mathbf{R}}$$

The vertical magnetic field required to balance toroidal forces

$$B_{\text{vert}} = \frac{F_{\text{hoop}} + F_{\text{tire}} + F_{1/R}}{2\pi R_0 I_p}$$

$$\frac{\epsilon}{4} B_{\theta a} \left( \ell_e + \ell_i + 2 + \frac{2\mu_0 \langle p \rangle}{B_{\phi a}^2} + \frac{B_{\phi a}^2 - \langle B_{\phi}^2 \rangle}{B_{\theta a}^2} \right)$$

$$= \frac{\mu_0 I_p}{4\pi R_0} \left( \ln \frac{8R}{a} - \frac{3}{2} + \frac{\ell_i}{2} + \beta_p \right)$$

Shafranov shift of the plasma center

$$\Delta = \frac{b^2}{2R_0} \left[ \left( \beta_p + \frac{\ell_i - 1}{2} \right) \left( 1 - \frac{a^2}{b^2} \right) + \ln \frac{b}{a} \right] - \frac{B_{\text{vert}}}{B_{\theta 1}(b)}$$

10.5 Plasma Para- and Dia-Magnetism

Global pressure balance equation in a screw pinch

$$\langle p \rangle = \frac{1}{2\mu_0} \left( B_{za}^2 - \langle B_z^2 \rangle + B_{\theta a}^2 \right)$$

can be rearranged to give

$$\beta_p = \frac{B_{za}^2 - \langle B_z^2 \rangle}{B_{\theta a}^2} + 1$$

Diamagnetic : $$\beta_p < 1$$  
Paramagnetic : $$\beta_p > 1$$
10.6 MHD Stability Limits

Using experimental data from a wide variety of tokamaks, empirical scalings for critical tokamak instabilities have been constructed. Units: current in MA, length in m, magnetic field in T, and density in \( n_{20} = n/10^{20} \).

(a) Beta limits (no plasma shaping)

\[
\beta_t \leq \beta_L \frac{I_a}{aB_\phi} \\
\beta_L = 0.028 \text{ Troyon kink limit - no wall}^{24} \\
\beta_L = 0.044 \text{ Sykes ballooning limit - no wall} \\
\beta_L = 0.06 \text{ kink - ideal conducting wall}
\]

(b) Definition of \( \beta_N \)^{26:347}

\[
\beta_N = \beta_t[\%] \frac{aB_\phi}{I_p[\text{MA}]}
\]

(c) The Greenwald (or Density) Limit^{26:377}

\[
n_{20} \leq n_G = \frac{I_p[\text{MA}]}{\pi a^2}
\]

10.7 Tokamak Heating and Current Drive

(a) Ohmic plasma heating

The neo-classical resistivity approximation is^{8:538}

\[
\eta_|| = \frac{1}{[1 - (r/R_0)^{1/2}]^2 \eta_{\text{Spitzer}}} 
\]

Plugging in for the current as \( J_|| = E_0/\eta_||^{8:539} \)

\[
P_\Omega = \left( \frac{5.6 \times 10^{-2}}{1 - 1.31 \epsilon^{1/2} + 0.46 \epsilon} \right) \left( \frac{R_0 (I[\text{MA}])^2}{a^2 \kappa T_{keV}^{3/2}} \right) \text{[MW]}
\]

(b) Neutral beam plasma heating^{26:246–248}

\[
P = \frac{m_b}{2 \pi e_0^2 m_e^2} ne^{4 \ln \Lambda} \left( \frac{2 m_e^{1/2} E_b}{3 (2 \pi)^{1/2} T_e^{3/2}} + \frac{m_b^{3/2}}{2^{3/2} m_i E_b^{1/2}} \right)
\]
where $E_b$ is the energy of the beam.

The critical beam energy when the ions and the electrons are heated equally by the beam is

$$E_c = 14.8 \frac{A_b}{A_i^{2/3}} T_e$$

### 10.7.1 Current Drive

(a) Inductive current

This current is driven via the central solenoid. The current distribution is calculated through the use of Faraday’s law and $J_{||} = E_0/\eta_{||}$. Total current normally has to be measured in order to normalize the distribution of current density.

(b) Bootstrap current

Bootstrap current is the self-generated current drive in the plasma from trapped and passing electrons in the plasma.

The exact form of the bootstrap current density is given by

$$j_B = -4.71q \left( \frac{R_0}{r} \right)^{1/2} \frac{T}{B_0} \left[ \frac{\partial n}{\partial r} + 0.04 \frac{n \partial T}{T \partial r} \right]$$

The total bootstrap fraction is given by

$$f_B \approx -1.18 \frac{\partial}{\partial r} (\ln n + 0.04 \ln T) / \frac{\partial}{\partial r} (\ln r B_\theta) \left( \frac{r}{R_0} \right)^{1/2} \beta_p \sim \epsilon^{1/2} \beta_p$$

(c) Neutral beam current drive

By positioning a neutral beam in the tangential direction, it is possible to drive both rotation and current. Neutral beam current drive efficiency scales as (at $E_b = 40A_b T_e$)

$$I[A]/P[W] \approx \frac{0.06T_e}{n_{20}RZ_b^2(1 - Z_b/Z_{eff})}$$

(d) Lower hybrid current drive

Currently one of the most used current drive mechanisms is the lower hybrid system. It launches a wave that Landau damps on the fast electron population and preferentially drives electrons in one direction.
There exists an accessibility condition for the waves which forces an increase in the launched $n_{||}$

$$n_{||}^2 > \left( S^{1/2} + \left| \frac{D^2}{P} \right|^{1/2} \right)^2$$

where $S$, $P$, and $D$ are defined in Chapter 8.

Because LHCD relies on Landau damping, there is an additional constraint on the $n_{||}$: Landau damping dominates at $n_{||}^2 \gtrsim 7.0/T_{keV}^{1/2}$.

(e) Fast Magnetosonic wave current drive

Allows peaked on-axis profiles and has the following current drive efficiency:

$$I[A]/P[W] = 0.025 \frac{T_{keV}}{n_{20} R_0}$$

## 10.8 Empirical Scaling Laws

### 10.8.1 Energy Confinement Time Scalings

Goldston auxiliary heated tokamak scaling ($l$ refers to the plasma size $\sim a$)

$$\tau_E \sim B_p^{2.18}/n_T$$

The ITER-89 L-Mode (ITER89-P)

$$\tau_E = 0.048 T_M^{0.85} R_0^{1.2} \kappa^{0.5} n_{20}^{0.1} B_0^{0.2} A^{0.5} P_M^{-0.5} [s]$$

The ITER-98 L-Mode

$$\tau_E = 0.023 T_M^{0.96} B_T^{0.03} n_{19}^{0.40} M^{0.20} R^{1.83} \epsilon^{-0.06} \kappa^{0.64} P_M^{-0.73} [s]$$

The ITER-98 (IPB98[y,2]); ELMy H-mode

$$\tau_E = 0.0562 T_M^{0.93} B_T^{0.15} n_{19}^{0.41} M^{0.19} R^{1.97} \epsilon^{0.58} \kappa^{0.78} P_M^{-0.69} [s]$$

Scaling for linear regime energy transport

$$\tau_E = 0.07 n_{20} q \kappa^{0.5} a R^2 [s]$$

Critical density of linear to saturated regime

$$n_{20} = 0.65 A_i^{0.5} B_T q^{-1} R^{-1}$$
10.8.2 Plasma Toroidal Rotation Scaling

Plasma toroidal rotation (Rice) scaling

\[ \Delta V_{\text{tor}} \propto \Delta W/I_p \]

The 2010 multi-machine scaling database found that (with \( v_a \) being the Alfvén speed)

\[ v/v_a = 0.65 \beta_1^{0.4} T_q^{2.3} j \]

where \( q_j = 2 \pi \kappa a^2 B/\mu_0 R I_p \)

10.8.3 L-H Mode Power Scalings

The ITPA empirical scaling law for the L to H mode transition power threshold

\[ P_{\text{L-H}} [\text{MW}] = 2.15 e^{\pm 0.107 n_20^{0.782 \pm 0.037} B_T^{0.772 \pm 0.031} a^{0.975 \pm 0.08} R^{0.999 \pm 0.101}} \]

10.9 Turbulence

Fundamental definitions

\[ L_n = n/\nabla n \quad L_T = T/\nabla T \]

\[ b = k_0^2 \rho_i^2 \quad \eta_j = L_{n_j}/L_{T,j} \]

\[ \epsilon = m_j v^2 / 2 T_j \]

The diamagnetic drift velocity

\[ v_d = \frac{\mathbf{B} \times \nabla p_j}{q_j n_j B^2} \]

Diamagnetic frequency

\[ \omega_{s,j} = \frac{k_0 T_j dn}{e B n dr} \]

Ion Larmor radius evaluated at the sound speed

\[ \rho_S \equiv c_s / \Omega_i \]

Normalized Larmor radius

\[ \rho_s \equiv \rho_S / a \]
10.9.1 General Drift Wave Turbulence

Mixing length estimate\textsuperscript{25}

\[ \tilde{n}_{\text{rms}}/n_0 \sim 1/k_\perp L_n \]

Density fluctuations and plasma potential correlation\textsuperscript{25}

\[ \tilde{n}/n_0 \approx \left( e\tilde{\phi}/kT_e \right) (1 - i\delta) \]

where \( \delta \) is the dissipation of the electron momentum to the background plasma.

Time averaged electrostatic turbulent flux of particles \( \tilde{\Gamma} \), momentum \( \vec{\mu} \), and heat \( \tilde{Q} \)\textsuperscript{25}

\[ \tilde{\Gamma} = -\left( \frac{\langle \tilde{n}\nabla\tilde{\phi} \rangle \times \vec{B}}{B^2} + \langle \tilde{n}\tilde{v}_\parallel \rangle B/B \right) \]

\[ \vec{\mu} = \left\langle \left( -\frac{\nabla\tilde{\phi}}{B^2} + \tilde{v}_\parallel B/B \right) \left( -\frac{\nabla\tilde{\phi}}{B^2} + \tilde{v}_\parallel B/B \right) \right\rangle \]

\[ \tilde{Q} \equiv \frac{5}{2} \tilde{n}T \left[ \frac{1}{T} \left( -\langle \tilde{T}\nabla\tilde{\phi} \rangle \times \vec{B}/B^2 + \langle \tilde{T}\tilde{v}_\parallel \rangle B/B \right) + \frac{1}{n} \left( -\langle \tilde{n}\nabla\tilde{\phi} \rangle \times \vec{B}/B^2 + \langle \tilde{n}\tilde{v}_\parallel \rangle B/B \right) \right] \]

where fluctuating values are marked by a tilde and \( \langle \rangle \) is a time average.

Time averaged momentum and and energy fluxes due to fluctuating magnetic fields\textsuperscript{25}

\[ \vec{\mu}^{EM} = \frac{\langle \vec{B}\vec{B} \rangle}{\mu_0\tilde{n}M_i} \]

\[ \tilde{Q}^{EM} = \frac{\langle \tilde{q}_\parallel \vec{e}\cdot\vec{B} \rangle}{B} \]

10.9.2 General Drift Turbulence Characteristics

Perpendicular drift wave turbulence is characterized by \( \rho_S \), with \( k_\parallel \ll k_\perp \) and \( k_\perp \rho_S \) depending on dissipative mechanism, linear free energy source, and nonlinear energy transfer.

Ion thermal gradient (ITG) turbulence occurs when \( \eta_i > \eta_{\text{crit}} \sim 1 \) and has the following approximate characteristics\textsuperscript{25}

\[ k_\perp \rho_s \sim 0.1 - 0.5 \]
10.9. TURBULENCE

(a) Fluctuations without parallel electron dissipation. (b) Fluctuations with finite electron dissipation. Figure from Tynan et al 2009. Copyright 1999 by the American Physical Society.

\[ \frac{R}{L_{T_i}} > \frac{R}{L_{T_i}}|_{\text{crit}} \sim 3 - 5 \]

\[ v_{ph} \sim v_{di} \]

Trapped electron mode (TEM) instabilities occurs at approximately \( k_{\perp} \rho_S \sim 1 \). At higher wavenumbers the TEM transitions into the electron thermal gradient (ETG) instability with \( \eta_e > \eta_{\text{crit}} \sim 1 \) and the following approximate characteristics\textsuperscript{25}

\[ k_{\perp} \rho_s \sim 1 - 10 \]

\[ \frac{R}{L_{T_e}} > \frac{R}{L_{T_e}}|_{\text{crit}} \sim 3 - 5 \]

10.9.3 Passing Particle Instabilities

In this section, it is assumed that \( k_{\parallel} v_{te} \gg \omega \gg k_{\parallel} v_{ti} \), such that electrons respond to the electrostatic potential. Also, the frequency of the magnetic curvature drifts is assumed to be \textsuperscript{26,422}

\[ \omega_{di} = 2 L_n \omega_{si} / R \ll \omega \]
The passing particle dispersion relation \(^{26:424}\):

\[
\left[ \rho^2 \frac{\partial^2}{\partial x^2} - \left( \frac{L_n/R}{b^{1/2}(T_e/T_i)q\Omega} \right)^2 \left( \frac{\partial}{\partial \theta} + ik_{\theta}sx \right) - \frac{2R/L_n}{(T_e/T_i)\Omega} \left( \cos \theta + \frac{i\sin \theta}{k_{\theta}} \frac{\partial}{\partial x} \right) - \left( \frac{\Omega - 1}{(T_e/T_i)\Omega + (1 + \eta_i)} + b \right)^2 \right] \tilde{\phi} = 0
\]

where \(x\) is the distance from the reference mode rational surface \(m = nq(r)\) and \(\tilde{\phi}\) is the perturbed electrostatic potential.

Ion thermal gradient (ITG, \(\eta_i\)) toroidal frequency \(^{26:428}\):

\[\omega_{\text{ITG}} \approx (\eta_i \omega_{\text{e}} \omega_{\text{di}})^{1/2}\]

ITG critical instability limit \(^{26:429}\):

\[\eta_{\text{ic}} = \begin{cases} 
1.2 & R/L_n < (R/L_n)_{\text{crit}} \\
4 \left(1 + \frac{T_i}{T_e}\right) \left(1 + 2s/q\right) R/L_n & R/L_n > (R/L_n)_{\text{crit}}
\end{cases}\]

where

\[(R/L_n)_{\text{crit}} = \frac{0.9}{(1+T_i/T_e)(1+2s/q)}\]

Electron thermal gradient (ETG, \(\eta_e\) mode) dispersion relation with \(T_i \approx T_e\) \(^{26:429}\):

\[-\frac{k^2}{\omega^2} \left(1 - \frac{\omega_{se}}{\omega}(1 + \eta_e)\right) + 1 + \frac{\omega_{se}}{\omega} = 0\]

If \(\eta_e \gg 1\) then there is an unstable mode with \(^{26:429}\) \(\omega \approx (-k^2 v_{te}^2 \eta_e \omega_{se})^{1/3}\)

### 10.9.4 Trapped Particle Modes

The collisionless trapped particle dispersion relation \(^{26:432}\):

\[
\frac{1}{\sqrt{2\epsilon}} \left( \frac{1}{T_i} + \frac{1}{T_e} \right) = \frac{1}{T_i} \left( \frac{\omega - \omega_{\text{ri}}}{\omega} \right) + \frac{1}{T_e} \left( \frac{\omega - \omega_{\text{de}}}{\omega} \right)
\]

where

\[\tilde{\omega}_{dj} = \frac{\omega_{dj}}{2} \left[ \left( \frac{v_{||}}{v_{\text{Thj}}} \right)^2 + \left( \frac{v_{\perp}}{2v_{\text{Thj}}} \right)^2 \right] \{ \cos \theta + \frac{k_r}{k_{\theta}} \sin \theta \}
\]

This dispersion relation gives rise to the trapped ion mode if \(\nu_{\text{eff}} = \nu_j/\epsilon > \omega_{dj}\) and has growth/frequency \(^{26:433 - 434}\):

\[
\omega = \frac{\sqrt{2\epsilon}}{1 + T_e/T_i} \omega_{se} - i \frac{\nu_i}{\epsilon} + i \frac{\epsilon^2}{(1 + T_e/T_i)^2} \nu_e \frac{\omega_{se}^2}{\nu_e}
\]
### 10.9. TURBULENCE

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Approximate range in $k_θ$ (cm$^{-1}$)</th>
<th>Approximate length scale(cm)</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density fluctuations ($\tilde{n}$)</td>
<td>&lt; 1</td>
<td>60</td>
<td>ITG</td>
</tr>
<tr>
<td>Temperature fluctuations ($\tilde{T}_e$)</td>
<td>&lt; 1</td>
<td>1</td>
<td>ITG</td>
</tr>
<tr>
<td>Flows, GAMs, ZF</td>
<td>&lt; 1</td>
<td>1</td>
<td>ITG</td>
</tr>
<tr>
<td>$n_e\tilde{T}_e$ cross phase</td>
<td>&lt; 1</td>
<td>1</td>
<td>ITG</td>
</tr>
<tr>
<td></td>
<td>&gt; 2</td>
<td>1-10</td>
<td>ITG, TEM</td>
</tr>
<tr>
<td></td>
<td>3-12</td>
<td>1</td>
<td>TEM</td>
</tr>
<tr>
<td></td>
<td>&gt; 20</td>
<td>1, 20</td>
<td>ETG</td>
</tr>
</tbody>
</table>

This mode has the largest imaginary part if $\nu_e \approx \epsilon^{3/2} \omega_{ce}$.

The TEM can be calculated due to the trapped particle dispersion relation. The mode is driven by trapped electron collisions and electron temperature gradients. 26: 434–435

If $\nu_{eff} \gg \omega_{ce}$ then the growth rate is

$$\gamma \approx \epsilon^{3/2} \frac{\omega_{ce}^2}{\nu_e} \eta_e$$
Chapter 11

Tokamak Edge Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

\( e \) is the elementary electric charge
\( q \) is the total particle charge
\( Z \) is the particle atomic (proton) number
\( T \) is the plasma temperature
\( n \) is the plasma number density
\( p \) is the plasma pressure
\( e \) and \( i \) refer to electrons and ions, respectively
\( a \) and \( R_0 \) are the minor and major radii of a toroidal plasma

11.1 The Simple Scrape Off Layer (SOL)

The simple SOL model describes 1D plasma flow from the core plasma to material boundary surfaces for limited or diverted plasma along the toroidal magnetic topology. By assuming a high degree of collisionality (\( \nu_* \)), fluid approximations for plasma flow are valid and the neoclassical effects on particle orbits due to toroidal magnetic topology can be safely ignored.

Parallel SOL connection length (rail/belt limiters, poloidal divertors) \(^{21:17}\)

\[ L_{\parallel} \approx \pi R q \]

Particle time in the SOL (simple 1D model) \(^{21:20}\)

\[ t_{\text{dwell}} \approx L_{\parallel}/c_s \]

SOL width (simple 1D model) \(^{21:23}\)

\[ \lambda_{\text{SOL}} \approx \left( D_{\perp} L_{\parallel}/c_s \right)^{1/2} \]

where \( D_{\perp} \) is the anomalous diffusion coefficient.
Force balance / pressure conservation in the SOL

\[ p_e + p_i + mnv^2 = \text{constant} \]

Plasma density variation along the SOL

\[ n(x) = \frac{n_0}{1 + M(x)^2} \]

where \( n_0 \) is the density at the ‘top’ of the SOL and \( M = v/c_s \) is the plasma mach number.

Electrons follow a Boltzmann distribution in the SOL

\[ n = n_0 \exp \left( \frac{eV}{T_e} \right) \]

SOL particle sources: ionization (i) and cross-field transport (t)

\[ S_p = S_{p,i} + S_{p,t} = n_{\text{plasma}} n_{\text{neutrals}} \langle \sigma v \rangle_i + D_{\perp} n / \lambda_{\text{SOL}}^2 \]

where \( \langle \sigma v \rangle_i \equiv \langle \sigma v \rangle_i(T_e, Z) \) is the ionization rate coefficient.

Particle flux density in the SOL at the sheath edge (se)

\[ \Gamma_{\text{se}} = \frac{1}{2} n_0 c_s \]

where \( n_0 \) is the density outside the pre-sheath.

Electric field through SOL required to satisfy the Bohm Criterion

\[ V_{\text{se}} = -0.7 \frac{T_e}{e} \]

Floating sheath voltage

\[ V_s = 0.5 \frac{T_e}{e} \ln \left[ 2\pi m_e \frac{n_e}{m_i} \left( Z + \frac{T_i}{T_e} \right) \right] \]

Debye sheath width

\[ \lambda_{\text{Debye}} \approx \left( \frac{\epsilon_0 T_e}{n_e e^2} \right)^{1/2} \]

11.2 Bohm Criterion

The Bohm Criterion is derived from conservation of energy \((1/2m_i v^2 = -eV)\) and particle conservation \((n_i v = \text{constant})\). In an unmagnetized plasma it sets the SOL plasma exit velocity into the sheath edge (se). In magnetized plasma, it sets the SOL plasma exit velocity parallel to the magnetic field, after which the ions become demagnetized and perpendicularly
11.3. A SIMPLE TWO POINT MODEL FOR DIVERTED SOLS

Diverted plasmas can obtain significant $\Delta T$ along the SOL, resulting in divertor temperatures less than 10 eV. The SOL can be approximated using a two point model: point 1 is the outboard midplane entrance to the SOL ("upstream" or "u") and point 2 is the divertor terminus of the SOL ("target" or "t"). It is assumed that upstream density, $n_u$, and the heat flux into the SOL, $q_{||}$, are control parameters; upstream and target temperatures, $T_u$ and $T_t$, as well as plasma density in front of the target, $n_t$, are subsequently determined.

11.3.1 Definitions

Dynamic and static pressure

$$p = nT (1 + M^2)$$

where

$$\begin{cases} 
M_u^2 \ll 1 \\
M_t^2 \approx 1 \text{ (Bohm Criterion)}
\end{cases}$$

Heat conduction parallel to magnetic field

$$q_{||, \text{ cond}} = -k_0 T^{5/2} \frac{dT}{dx}$$

where

$$\begin{cases} 
k_{e,0} \approx 2000 \ \text{[W m}^{-1} \text{eV}^{7/2}] \\
k_{i,0} \approx 60 \ \text{[W m}^{-1} \text{eV}^{7/2}]
\end{cases}$$

Sheath heat flux transmission coefficient at a biased surface

$$\gamma = 2.5 \frac{T_i}{ZT_e} - \frac{eV}{T_e} + 2 \left[ 2\pi \frac{m_e}{m_i} \left( Z + \frac{T_i}{T_e} \right) \right]^{-1/2} \exp \left( \frac{eV}{T_e} \right) + \chi_i$$

where $T_e \neq T_i$, $\chi_i$ is the electron-ion recombination energy, and no secondary electrons emitted.
11.3.2 Fundamental Relations

SOL pressure conservation \[^{21}:^{224}\]

\[ 2n_tT_t = n_uT_u \]

SOL power balance \[^{21}:^{224}\]

\[ T_u^{7/2} = T_t^{7/2} + \frac{7q_||L}{2k_0} \]

SOL heat flux limited to sheath heat flux \[^{21}:^{224}\]

\[ q_|| = \gamma n_tT_t c_s \approx 7 \quad \text{(D-D plasma, floating surface)} \]

11.3.3 Consequences

Upstream SOL temperature \[^{21}:^{226}\]

\[ T_u \approx \left( \frac{7q_||L}{2k_0} \right)^{2/7} \text{ assuming that } T_t^{7/2} \ll T_u^{7/2} \]

\[ \rightarrow T_u \text{ is independent of } n_u \]
\[ \rightarrow T_u \text{ is insensitive to parameter changes due to the } 2/7 \text{ power} \]
\[ \rightarrow q_|| \text{ is extremely sensitive to } T_u \text{ due to the } 7/2 \text{ power} \]

Target SOL temperature \[^{21}:^{227}\]

\[ T_t \approx \frac{2m_i}{\gamma^2 e^2} \frac{q_||^{10/7}}{(Lk_0)^{4/7}n_u^2} \]

\[ \rightarrow T_t \text{ is proportional to } \frac{1}{n_u} \]

Target SOL density \[^{21}:^{227}\]

\[ n_T = \frac{n_u^3}{q_||^2} \left( \frac{7q_||L}{2k_0} \right)^{6/7} \frac{\gamma^2 e^2}{4m_i} \]

\[ \rightarrow n_T \text{ is proportional to } n_u^3 \]
Chapter 12

Tokamak Fusion Power

In this chapter, all units are SI with the exception of temperature and energy, which are defined in the historical units of eV (electron-volts).

\( n \) is the plasma density; \( n_{20} = n/10^{20} \); \( n_0 = n_e \approx n_i \)
\( T \) is the plasma temperature; \( T_{\text{keV}} = T \) in units of kiloelectron-volts
\( p \) is the plasma pressure
\( \nu \) is the radial profile peaking factor
\( P \) is a power density
\( S \) is a total power
\( U \) is a total energy
\( E \) is the nuclear reaction energy gain
\( e \) is the elementary electric charge
\( q \) is the total particle charge
\( Z \) is the particle atomic (proton) number
\( e \) and \( i \) subscripts refer to electrons and ions, respectively
\( D \) and \( T \) refer to deuterium and tritium, respectively
\( \kappa \) is the plasma elongation; \( \kappa = b/a \)
\( a, b, \) and \( R_0 \) are the 2 minor and major radii of a toroidal plasma
\( V \) is the volume of the plasma

12.1 Definitions

Fusion power density \( 26:8 \)

\[
P_{\text{fusion}} = n_D n_T \langle \sigma v \rangle_{DT} E_{\text{fusion}} = \frac{1}{4} n_e^2 \langle \sigma v \rangle_{DT} E_{\text{fusion}}
\]

Fusion power \( 26:22 \)

\[
S_{\text{total}} = \frac{\pi}{2} E_{\text{fusion}} \int_{\text{plasma cross section}} n^2 \langle \sigma v \rangle_{DT} R dS
\]

\[
= \frac{0.15}{2\nu + 1} Rab \left( \frac{n}{10^{20}} \right)^2 T_{\text{keV}}^2 \quad [\text{MW}]
\]
where it has been assumed that:

- **Pressure profile:**
  \[ nT = \hat{n}T \left( 1 - \frac{r^2}{\tilde{a}^2} \right)^\nu \]

- **Reaction rate:**
  \[ \langle \sigma v \rangle_{DT} \approx 1.1 \times 10^{-24} T_{keV}^{-2} \]

- **Plasma cross section:**
  \[ \tilde{a} = (ab)^{1/2} \]

**Alpha power density**
\[ P_\alpha = n_D n_T \langle \sigma v \rangle_{DT} E_\alpha = \frac{1}{4} n_e^2 \langle \sigma v \rangle_{DT} E_\alpha \approx (1/5) P_{\text{fusion}} \]

**Neutron power density**
\[ P_{\text{neutron}} = n_D n_T \langle \sigma v \rangle_{DT} E_{\text{neutron}} = \frac{1}{4} n_e^2 \langle \sigma v \rangle_{DT} E_{\text{neutron}} \approx (4/5) P_{\text{fusion}} \]

**Ohmic heating power density**
\[ P_{\text{ohmic}} \approx \eta J^2_{\text{plasma}} \]

**Stored energy in confined plasma**
\[ W = \int 3nT \, d\tau = 3\langle nT \rangle V \]

**Definition of energy confinement time**
\[ \tau_E = \frac{\text{Stored energy in the confined plasma}}{\text{Power lost from the confined plasma}} = \frac{W}{S_{\text{loss}}} \]

**Power loss from a confined plasma due to conduction**
\[ P_{\text{conduction}} = \frac{3nT}{\tau_E} \]

**Power loss from a confined plasma due to bremsstrahlung radiation**
\[ P_{\text{bremsstrahlung}} \approx (5.35 \times 10^{-37}) Z^2 n_e n_i T_{keV}^{1/2} \quad [W \, m^{-3}] \]

## 12.2 Power Balance in a D-T Fusion Reactor

Confined fusion plasmas are not in thermal equilibrium, and, therefore, power must be balanced in a steady-state tokamak reactor. Power that is lost from the confined plasma due to conduction, radiation and other mechanisms must be continuously replenished by alpha particle and auxiliary heating mechanisms.

\[ 0 = (P_{\alpha} + P_{\text{auxilliary}}) - (P_{\text{conduction}} + P_{\text{bremsstrahlung}} + \ldots) \]

where
\[ P_{\text{auxilliary}} = P_{\text{ohmic}} + P_{\text{ICH}} + P_{\text{ECH}} + P_{\text{neutral beam}} + \ldots \]
12.2.1 Impurity Effects on Power Balance

The fractional impurity densities \( f_j = n_j/n_0 \) in the plasma core cause:

(a) Modified quasi-neutrality balance \(^{26,36}\)
\[
n_e = n_D + n_T + \sum_j Zn_j
\]

(b) Increased radiated power loss \(^3\)
\[
P_{\text{bremsstrahlung}} \approx (5.35 \times 10^{-37})n_e^2T_{\text{keV}}^{1/2}Z_{\text{eff}} [\text{W m}^{-3}]
\]

(c) Dilution of fusion fuel \(^3\)
\[
P_{\alpha} = \frac{1}{4}n_e^2(1 - \sum_j f_j Z_j)^2 \langle \sigma v \rangle E_\alpha
\]

12.2.2 Metrics of Power Balance

The physics gain factor for D-T plasma \(^{26,12}\)
\[
Q_{\text{phys}} = \frac{\frac{1}{4}n_e^2 \langle \sigma v \rangle E_{\text{fusion}} \cdot V_{\text{plasma}}}{P_{\text{heating}}} = \frac{5P_\alpha}{P_{\text{heating}}}
\]

where

(a) \( Q_{\text{phys}} = 1 \) is break even

(b) \( Q_{\text{phys}} > 5 \) is a burning plasma

(c) \( Q_{\text{phys}} = \infty \) is an ignited plasma

The engineering gain factor \(^3\)
\[
Q_{\text{eng}} = \frac{P_{\text{out}}}{P_{\text{in}}}
\]

12.3 The Ignition Condition (or Lawson Criterion)

The ignition condition describes the minimum values for density \((n)\), temperature \((T)\), and energy confinement time \((\tau_E)\) that are required for a confined plasma to reach ignition. Ignition is defined as \( P_{\alpha} > P_{\text{loss}} \), where \( P_{\text{auxilliary}} = 0. \(^{26,10-15}\) For a given temperature, \( T \), the following equations describe the minimum \( n\tau_E \) required to reach ignition under different assumptions:
(a) $P_{\alpha} = P_{\text{conduction}}^{26:10-11}$

$$n \tau_E = \frac{12kT}{\langle \sigma v \rangle E_\alpha}$$

Using $\langle \sigma v \rangle_{DT} \approx 1.1 \times 10^{-24} T_{\text{keV}}^2$ and $E_\alpha = 3.5 \text{ MeV}^3$:

$$nT \tau_E \gtrsim 3 \times 10^{21} \text{ m}^{-3} \text{ keV s}$$

(b) $P_{\alpha} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}^{3}$:

$$n \tau_E = \frac{12kT}{\langle \sigma v \rangle E_\alpha - 2.14 \times 10^{-36} T_{\text{keV}}^{1/2}}$$

(c) $P_{\alpha} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}$ with alpha impurities $f_\alpha = n_\alpha/n_e$:

$$n \tau_E = \frac{12kT}{(1 - 2f_\alpha)^2 \langle \sigma v \rangle E_\alpha - (1 + 2f_\alpha)2.14 \times 10^{-36} T_{\text{keV}}^{1/2}}$$

(d) $P_{\alpha} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}$ with impurity densities $f_j = n_j/n_e$:

$$n \tau_E = \frac{12kT}{(1 - \sum_j f_j Z_j)^2 \langle \sigma v \rangle E_\alpha - (2.14 \times 10^{-36}) Z_{\text{eff}} T_{\text{keV}}^{1/2}}$$
Chapter 13

Tokamaks of the World

- Comprehensive list of all major tokamaks with parameters
  www.tokamak.info

- ASDEX-U (Garching, Germany)
  http://www.ipp.mpg.de/ippcms/eng/for/projekte/asdex/techdata.html

- Alcator C-Mod (Cambridge, USA)
  http://www.psfc.mit.edu/research/alcator/

- DIII-D (San Diego, USA)
  https://fusion.gat.com/global/Home

- EAST (Hefei, China)
  http://english.hf.cas.cn/1c/1p/east

- FTU (Frescati, Italy)
  http://www.efda.org/eu_fusion_programme/machines-ftu_i.htm

- Ignitor (Kurchatov, Russia)
  http://www.frascati.enea.it/ignitor

- ITER (Cadarache, France)
  http://www.iter.org/mach

- JET (Culham, United Kingdom)
  http://www.jet.efda.org/jet/jets-main-features

- J-TEXT (Wuhan, China)

- JT-60SA (Naka, Japan)
  http://www-jt60.naka.jaea.go.jp/english/figure-E/html/figureE_jt60sa_5.html

- KSTAR (Daejeon, S. Korea)

- NSTX-U (Princeton, USA)
  http://nstx-u.pppl.gov/
• SST-1 (Gandhinagar, India)
  http://www.ipr.res.in/sst1/SST1parameters.html

• T-10 (Kurchatov, Russia)

• T-15U (Kurchatov, Russia)

• TCV (Lausanne, Switzerland)
  https://crppwww.epfl.ch/tcv

• TEXTOR (Jülich, Germany)
  http://www2.fz-juelich.de/ief/ief-4//textor_en

• TFTR (Princeton, USA)
  http://w3.pppl.gov/tftr/info/tftrparams.html

• Tore Supra (Cadarache, France)
  http://www-fusion-magnetique.cea.fr/gb/index.html
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aMagnet coils are conducting unless denoted as superconducting (SC)
bPlasma configuration: L = limited; SN = diverted single null; DN = diverted double null
cPlasma Facing Components: Be = beryllium; C = CFC/graphite; Li = lithium; M = molybdenum; SS = stainless steel; W = tungsten
dProposed; construction not begun
eConstruction begun; first plasma predicted 2019
fConstruction begun: first plasma predicted 2014
gConstruction begun: first plasma predicted 2012
hDecommissioned in 1997
References


[3] Calculated, derived or produced by the authors of the Magnetic Fusion Energy Formulary explicitly for this work.


Evaluated Nuclear Data File ENDF/B-VII.0.


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