Fokker-Planck Equation for Moderately Coupled Plasmas

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A Fokker-Planck equation has been generalized to treat large-angle as well as small-angle binary collisions in plasmas. For moderately coupled plasmas [$2 \lesssim$ Coulomb logarithm $(\ln\Lambda_b) \lesssim 10$], calculations have been made of the relaxation rates and transport coefficients. In general they differ from the standard (Braginskii) results by terms of order $1/\ln\Lambda_b$. Using a modified collision operator, we obtain a new vector potential that has a direct and practical connection to the Rosenbluth potentials. In addition, we calculate a reduced electron-ion collision operator that, for the first time, manifests $1/\ln\Lambda_b$ corrections.

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The Fokker-Planck equation, which was originally derived to treat the Brownian motion of molecules [1,2], has been widely used to evaluate the collision term of the Boltzmann equation for describing small-angle binary collisions of the inverse-square type of force. In stellar dynamics [3], Chandrasekhar first discussed this theory for stochastic effects of gravity. The applications of this equation to classical plasma physics were first treated by Landau [4] and Spitzer [5], as well as Cohen, Spitzer, and Routly [6], and an elegant mathematical treatment was completed by Rosenbluth, MacDonald, and Judd [7]. Their treatments, as well as those of other workers [8-12], are based on the assumption that the Coulomb logarithm ($\ln\Lambda_b$), which is a measure of the importance of small-angle binary collisions relative to large-angle scattering, is of order 10 or greater. Terms smaller by the factor of the Coulomb logarithm are neglected, i.e., large-angle scattering is ignored. The conventional Fokker-Planck equation, applicable to weakly coupled plasmas ($\ln\Lambda_b \gtrsim 10$), is therefore only accurate to within an order of the Coulomb logarithm [4-12]. However, there is a large class of plasmas for which the approximation is invalid [13]: strongly coupled plasmas at one extreme ($\ln\Lambda_b \lesssim 1$) [14-17] and moderately coupled ones in the intermediate regime ($2 \lesssim \ln\Lambda_b \lesssim 10$) [15, 18-28]. It is to moderately coupled plasmas, as exemplified by short-pulse laser plasmas [19-22], inertial confinement fusion plasmas [23], x-ray laser plasmas [26,27], and the solar core [28], to which our modifications of the Fokker-Planck equation are directed. As discussed in detail elsewhere [29], our modifications consist of retaining the third-order term and parts of the second-order term [30], both of which are usually discarded [3-12] in the Taylor expansion of the collision operator. (Fourth-, fifth-, sixth-, and higher-order terms in the expansion will be ignored since they are smaller than the third-order term by factors of 8, 80, 960, ..., respectively.) After presenting some basic properties of the collision operator, we use it to calculate a reduced electron-ion collision operator, relaxation rates, and first-order transport coefficients.

The Boltzmann equation for the rate of change of the test particle (subscript or superscript t) distribution is

$$\frac{\partial f_t}{\partial \tau}+\mathbf{v}\cdot\frac{\partial f_t}{\partial x}+\mathbf{a}\cdot\frac{\partial f_t}{\partial \mathbf{v}}=\left(\frac{\partial f_t}{\partial \tau}\right)_{\text{coll}} . \tag{1}$$

$(\partial f_t/\partial\tau)_{\text{coll}}$ is the collision operator and represents the time-rate-of-change of f_t due to collisions with the field particles (subscript or superscript f). Its Taylor expansion [8,11] is written as

$$\left(\frac{\partial f_t}{\partial \tau}\right)_{\text{coll}}=-\frac{\partial}{\partial v_i}(f_t\langle\Delta v_i\rangle^{t/f})+\frac{1}{2}\frac{\partial^2}{\partial v_i\partial v_j}(f_t\langle\Delta v_i\Delta v_j\rangle^{t/f})-\frac{1}{6}\frac{\partial^3}{\partial v_i\partial v_j\partial v_k}(f_t\langle\Delta v_i\Delta v_j\Delta v_k\rangle^{t/f}) , \tag{2}$$

where the v_i, v_j, and v_k represent the components of the test-particle velocity in Cartesian coordinates. In our calculation, we follow the conventions of Rosenbluth, MacDonald, and Judd [7] and Trubnikov [8]:

$$\langle\Delta v_i\rangle^{t/f}=-L^{t/f}\left(\frac{m_t+m_f}{m_f}\right)\frac{\partial}{\partial v_i}H(\mathbf{v}) , \tag{3}$$

$$\langle\Delta v_i\Delta v_j\rangle^{t/f}=-2L^{t/f}\frac{\partial^2}{\partial v_i\partial v_j}G(\mathbf{v})+\frac{L^{t/f}}{\ln\Lambda_b}\left[3\frac{\partial^2}{\partial v_i\partial v_j}G(\mathbf{v})-\delta_{ij}H(\mathbf{v})\right] , \tag{4}$$

$$\langle\Delta v_i\Delta v_j\Delta v_k\rangle^{t/f}=4L^{t/f}\frac{1}{\ln\Lambda_b}\left(\frac{m_f}{m_t+m_f}\right)\frac{\partial^2}{\partial v_i\partial v_j}\mathbf{\Phi}(\mathbf{v}) . \tag{5}$$

$L^{t/f}=(4\pi e_t e_f/m_t)^2\ln\Lambda_b$, where $\ln\Lambda_b=\ln(\lambda_D/p_\perp)$, λ_D is the Debye length of the field particles; $p_\perp=e_te_f/m_ru^2$ is the impact parameter for 90° scattering, with m_r the reduced mass, $e_t(e_f)$ the test (field) charges, and $u=|\mathbf{v}-\mathbf{v}'|$ the relative velocity; and $m_t(m_f)$ is the test (field) particle mass. In addition,

$$H(\mathbf{v})=-\frac{1}{4\pi}\int\frac{f_f(\mathbf{v}')}{|\mathbf{u}|}d\mathbf{v}' , \tag{6}$$

$$G(\mathbf{v})=-\frac{1}{8\pi}\int|\mathbf{u}|f_f(\mathbf{v}')d\mathbf{v}' , \tag{7}$$

and

$$\mathbf{\Phi}(\mathbf{v}) = -\frac{1}{32\pi}\int \mathbf{u}|\mathbf{u}|f_f(\mathbf{v}')d\mathbf{v}' . \tag{8}$$

H and G, which appear in Eqs. (3) and (4), are potentials defined by Rosenbluth *et al.* [7,12]. $\mathbf{\Phi}$ is a new vector potential that derives from retaining the third term in Eq. (2). In Eqs. (4) and (5), the factors multiplied by $1/\ln\Lambda_b$ are a direct consequence of our third-order expansion. In contrast to $\langle \Delta v_i \rangle^{t/f}$ and $\langle \Delta v_i \Delta v_j \rangle_{\perp}^{t/f}$ which represent the effects of small-angle collisions [8,11], $\langle \Delta v_i \Delta v_j \rangle_{\parallel}^{t/f}$ and $\langle \Delta v_i \Delta v_j \Delta v_k \rangle^{t/f}$ mainly represent the effects of large-angle scattering.

The vector potential $\mathbf{\Phi}$ has the following useful properties:

$$\nabla_v^2 \nabla_v^2 \nabla_v \cdot \mathbf{\Phi}(\mathbf{v}) = f_f(\mathbf{v}) , \tag{9}$$

$$\nabla_v^2 \nabla_v^2 \mathbf{\Phi}(\mathbf{v}) = 0 , \tag{10}$$

$$\begin{aligned} \nabla_v^2 \nabla_v \cdot \mathbf{\Phi}(\mathbf{v}) &= \nabla_v^2 G(\mathbf{v}) \\ &= H(\mathbf{v}) . \end{aligned} \tag{11}$$

Equation (11) is utilized to obtain the reduced electron-ion collision operator in the approximation where $m_e/m_{\text{ion}} \simeq 0$:

$$C_{e\text{-ion}}(f_e,f_{\text{ion}}) = A\frac{\partial}{\partial v_i}\left[\left(1-\frac{5}{6\ln\Lambda_b}\right)\mathrm{V}_{ij}\frac{\partial}{\partial v_j}f_e + \frac{1}{\ln\Lambda_b}\left(\frac{4}{3}\frac{v_i}{v^3} + \frac{\delta_{ij}}{v}\frac{\partial}{\partial v_j} + \frac{\mathrm{V}_{ijk}}{6}\frac{\partial^2}{\partial v_j \partial v_k}\right)f_e\right] , \tag{12}$$

where $A = 2\pi n_e Z^2 e^4 \ln\Lambda_b / m_e^2$, V_{ij} is the conventional diffusion tensor [8,11,29,31,32] in velocity space,

$$\mathrm{V}_{ij} = \frac{v^2\delta_{ij} - v_i v_j}{v^3} , \tag{13}$$

and the new terms, including the third-rank tensor,

$$\mathrm{V}_{ijk} = \frac{v_i}{v}\delta_{jk} + \frac{v_j}{v}\delta_{ki} + \frac{v_k}{v}\delta_{ij} - \frac{v_i v_j v_k}{v^3} , \tag{14}$$

are mainly associated with large-angle scattering. Once again the terms in Eq. (12) with coefficient $1/\ln\Lambda_b$ are a consequence of this new expansion.

For the purpose of illustrating the effects of these modifications, we show in Tables I and II some physical quantities of interest. The column labeled Conventional (Modified) denotes the results without (with) the modifications [5,8,9,12,24,29,31-34]. To calculate the relaxation rates of Table I, we used a delta function for the test-particle velocity and a Maxwellian distribution for the field particles. We find that even with the inclusion of all higher terms, the slowing down rate is unmodified from the conventional form [8,12]. As best as we can tell, this seems not to have been previously recognized since other workers indicated $\ln\Lambda_b$ must be 10 or greater for its application. In contrast to this, the energy loss rate and 90° deflection rate both manifest $1/\ln\Lambda_b$ corrections. To the best of our knowledge, this is the first time these corrections have been calculated. For the energy loss rate, it is in part utilized in estimating the energy loss for 3.5 MeV α's, 1.01 MeV tritons, 0.82 MeV ^{3}He, and fast electrons in inertial confinement fusion plasmas [35]. In calculating the 90° deflection rate, the vector potential [Eq. (8)] is utilized. We find an increase in this rate over the standard (Braginskii) form; this means that the mean free path, used extensively in discussions of high-gradient scale-length plasmas [19-27,31-34], will be decreased.

To calculate the electron thermal conductivity with the new collision operator [Eq. (12)], the electron distribution function was expanded as a first-order Legendre polynomial, i.e., $f_e = f_0 + f_1\cos\theta$, where θ is the angle between $\mathbf{v}$ and the direction of the heat flow, and a high-Z

TABLE I. The lists of the modified (conventional) relaxation rates.

Relaxation rates	Conventional	$\ln\Lambda_b \gtrsim 10$ restriction[a]	Modified
Slowing Down $(\nu_s^{t/f})$ [b,c]	$\left(1+\frac{m_t}{m_f}\right)\mu\nu_0$	No[d]	Unchanged from conventional
90° Deflection $(\nu_D^{t/f})$ [b,c]	$2\left(\mu+\mu'-\frac{\mu}{2x^{t/f}}\right)\nu_0$	Yes	$2\left[1+\frac{1}{2\ln\Lambda_b}\left(\frac{m_f-m_t}{m_f+m_t}\right)\right]\left(\mu+\mu'-\frac{\mu}{2x^{t/f}}\right)\nu_0$ [e]
Energy loss $(\nu_E^{t/f})$ [b,c]	$2\left(\frac{m_t}{m_f}\mu-\mu'\right)\nu_0$	Yes	$2\left[\frac{m_t}{m_f}\mu-\mu'+\frac{1}{\ln\Lambda_b}(\mu+\mu')\right]\nu_0$

[a]This condition applies to the conventional results only.
[b]$\nu_0 = \sqrt{2}\pi e_t^2 e_f^2 n_f \ln\Lambda_b / \sqrt{m_t} E_t^{3/2}$—the "basic relaxation rate."
[c]$\mu = 2\int_0^{x^{t/f}} e^{-\xi}\sqrt{\xi}d\xi/\sqrt{\pi}$—the Maxwell integral, and μ' its first derivative.
[d]As best as we can tell, this was not known since it is stated [12] that $\ln\Lambda_b \gtrsim 10$.
[e]$\nu_D^{t/f} \simeq [\langle(\Delta \mathbf{v}_\perp)^2\rangle/v^2] - [\langle(\Delta \mathbf{v}_\perp)^2\Delta \mathbf{v}\rangle/v^3]$.

TABLE II. The lists of the modified (conventional) transport coefficients.

Transport coefficients	Conventional	$\ln\Lambda_b \gtrsim 10$ restriction[a]	Modified
Electron electric conductivity ($\sigma_\parallel$)[b]	$\frac{32 n_e e^2}{3\pi m_e v_{ei}}$	Yes	$\frac{32 n_e e^2}{3\pi m_e (1+1/6\ln\Lambda_b) v_{ei}}$
Electron thermal conductivity ($\kappa_\parallel$)[b]	$\frac{16\sqrt{2\pi} n_e k T_e}{3 m_e v_{ei}}$	Yes	$\frac{16\sqrt{2\pi} n_e k T_e}{3 m_e (1+1/6\ln\Lambda_b) v_{ei}}$
Light collision damping rate (ν)[c]	$\frac{\omega_{pe}^2}{\omega^2}\frac{8\pi}{3}A\bar{f}_0$	Yes	$\frac{\omega_{pe}^2}{\omega^2}\frac{8\pi}{3}\left(1+\frac{1}{6\ln\Lambda_b}\right)A\bar{f}_0$

[a]This condition applies to the conventional results only.
[b]$v_{ei} = 4\sqrt{2\pi} Z^2 e^4 n_e \ln\Lambda_b / 3\sqrt{m_e}(kT_e)^{3/2}$—the "basic electron-ion collision rate," and here $v_e^2 = kT_e/m_e$ is assumed [12].
[c]$\bar{f}_0 = -\int_0^\infty dv\, g(v)(\partial f_0/\partial v)$, where $g(v) = \{1+[2A/v^3\omega]^2\}^{-1} \simeq 1$ due to $v_{ei}/\omega \ll 1$ [24], and ω is the frequency of the laser light wave.

limit for the ion charge has been assumed (Lorentz-gas model). In addition, the Boltzmann equation was also linearized. (A similar procedure was used in the calculation of other transport coefficients listed in Table II.) In this approximation, $1/6\ln\Lambda_b$ corrections are evident. The fact that these corrections are much smaller than the $\ln\Lambda_b$ corrections of the energy loss and deflection rate is, we conjecture, due to the linearization of the Boltzmann equation and retention of only the first-order correction in the electron distribution function. Therefore, in future work it will be important to include higher-order terms of the electron distribution function as well as retaining the nonlinearities of the Boltzmann equation. Such transport coefficients could then be applied to a variety of plasmas, such as short-pulse laser plasmas [19–22], x-ray laser plasmas [26,27], inertial confinement fusion plasmas [23], and the solar core [28].

In summary, we have modified the standard Fokker-Planck operator for Coulomb collisions by including terms that are directly associated with large-angle scattering. The procedure allows us to effectively treat plasmas for which $\ln\Lambda_b \gtrsim 2$, i.e., for moderately coupled plasmas. Precise calculations of some relaxation rates and rough estimates of electron transport coefficients were made, and, in most cases, they differed from Braginskii's and Trubnikov's results by terms of order $1/\ln\Lambda_b$. However, in the limit of large $\ln\Lambda_b$ ($\gtrsim 10$), these results reduce to the standard (Braginskii) form [5,8,9,12]. In addition, we have calculated a reduced electron-ion collision operator that for the first time manifests the $1/\ln\Lambda_b$ corrections.

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[1] A. D. Fokker, Ann Phys. (Leipzig) **43**, 812 (1914).
[2] M. Planck, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl, 324 (1916).
[3] S. Chandrasekhar, Principle of Stellar Dynamics (Dover, New York, 1943).
[4] L. D. Landau, Zh. Eksp. Teor. Fiz. **7**, 203 (1937).
[5] L. Spitzer, Mon. Not. R. Astron. Soc. **100**, 396 (1940).
[6] R. S. Cohen, L. Spitzer, and P. McR. Routly, Phys. Rev. **80**, 230 (1950).
[7] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[8] B. Trubnikov, *Review of Plasma Physics 1* (Consultants Bureau, New York, 1965).
[9] S. I. Braginskii, *Review of Plasma Physics 1* (Ref. [8]).
[10] D. J. Sigmar and G. Joyce, Nucl. Fusion **11**, 447 (1971).
[11] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964); I. P. Shkrarofsky, T. W. Johnston, and M. P. Backynski, *The Particle Kinetics of Plasmas* (Addison-Wesley, Reading, MA, 1966).
[12] D. L. Book, *NRL Plasma Formulary* (U.S. Naval Research Laboratory, Washington, DC, 1990).
[13] R. D. Petrasso, Nature (London) **343**, 21 (1990).
[14] "Strongly Coupled Plasma Physics," *Proceedings of Yamada Conference,* edited by S. Ichimaru (North-Holland, Amsterdam, 1990).
[15] R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley, Palo Alto, 1990).
[16] S. L. Gilbert, J. J. Bollinger, and D. J. Wineland, Phys. Rev. Lett. **60**, 2022 (1988); J. J. Bollinger and D. J. Wineland, Sci. Am. **262**, No. 1, 124 (1990).
[17] E. Y. Andrei, S. Yücel, and L. Menna, Phys. Rev. Lett. **67**, 3704 (1991).
[18] F. J. Rogers, H. E. Dewitt, and D. B. Boercker, Phys. Lett. **82A**, 331 (1981).
[19] D. Riley, L. A. Gizzi, F. Y. Khattak, A. J. Mackinnon, S. M. Viana, and O. Willi, Phys. Rev. Lett. **69**, 3739 (1992).
[20] J. Denavit, Phys. Rev. Lett. **69**, 3052 (1992).
[21] H. M. Milchberg, R. R. Freeman, S. C. Davey, and R. M. More, Phys. Rev. Lett. **61**, 2364 (1988).
[22] A. Ng, D. Parfeniuk, P. Celliers, L. DaSilva, R. M. More, and Y. T. Lee, Phys. Rev. Lett. **57**, 1595 (1986).
[23] J. D. Lindl, R. L. McCrory, and E. M. Campbell, Phys. Today **45**, No. 9, 32 (1992), and references therein.
[24] W. L. Kruer, *The Physics of Laser Plamsa Interactions*

(Addison-Wesley, Reading, MA, 1988).
[25] Y. T. Lee and R. M. More, Phys. Fluids **27**, 1273 (1984).
[26] D. L. Matthews, P. L. Hagelstein, M. D. Rosen, M. J. Eckart, N. M. Ceglio, A. U. Hazi, H. Medecki, B. J. MacGowan, J. E. Trebes, B. L. Whitten, E. M. Campbell, C. W. Hatcher, A. M. Hawryluk, R. L. Kauffman, L. D. Pleasance, G. Rambach, J. H. Scofield, G. Stone, and T. A. Weaver, Phys. Rev. Lett. **54**, 110 (1985); and references therein; (private communication with D.L.M.).
[27] M. D. Rosen, P. L. Hagelstein, D. L. Matthews, E. M. Campbell, A. U. Hazi, B. L. Whitten, B. MacGowan, R. E. Turner, R. W. Lee, G. Charatis, Gar. E. Busch, C. L. Shepard, and P. D. Rockett, Phys. Rev. Lett. **54**, 106 (1985).
[28] J. N. Bahcall, *Neutrino Astrophysics* (Cambridge Univ. Press, New York, 1989), and references therein.
[29] C. K. Li, R. D. Petrasso, and L. Wang (to be published).
[30] J. Hubbard, Proc. R. Soc. London A **261**, 371 (1961); E. C. Shoub, Astrophys. J. **389**, 558 (1992).
[31] E. M. Epperlein, G. J. Rickard, and A. R. Bell, Phys. Rev. Lett. **61**, 2453 (1988).
[32] G. J. Rickard, A. R. Bell, and E. M. Epperlein, Phys. Rev. Lett. **62**, 2687 (1989).
[33] J. P. Matte, T. W. Johnston, J. Delettrez, and R. L. McCrory, Phys. Rev. Lett. **53**, 1461 (1984); D. Shavarts, J. Dellettrez, R. L. McCrory, and C. P. Verdon, Phys. Rev. Lett. **47**, 247 (1981).
[34] C. K. Li and R. D. Petrasso, Proceedings of Short Wavelength V: Physics with Intense Laser Pulses, San Diego, CA, 29-31 March 1993 (to be published).
[35] C. K. Li and R. D. Petrasso, preceding Letter, Phys. Rev. Lett. **70**, 3059 (1993).