Spatial dispersion in metamaterials with negative
dielectric permittivity and
its effect on surface waves

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The effect of spatial dispersion on the electromagnetic properties of a metamaterial consisting of a three-
dimensional mesh of crossing metallic wires is reported. The effective dielectric permittivity tensor \( \varepsilon_r(\omega,k) \)
of the wire mesh is calculated in the limit of small wavenumbers. The procedure for extracting the spatial
dispersion from the \( \omega \) versus \( k \) dependence for electromagnetic waves propagating in the bulk of the
metamaterial is developed. These propagating modes are identified as similar to the longitudinal (plasmon)
and transverse (photon) waves in a plasma. Spatial dispersion is found to have the most dramatic effect on
the surface waves that exist at the wire mesh–vacuum interface.© 2006 Optical Society of America

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Metamaterials with exotic electromagnetic properties such as negative effective dielectric permittivity
\( \varepsilon \) and/or magnetic permeability \( \mu \) have received significant attention\(^{1–4} \) because of applications in
superlensing,\(^{5} \) particle accelerators,\(^{6,7} \) and microwave sources.\(^{8} \) One such metamaterial is a three-
dimensional (3D) mesh of orthogonal crossing metallic wires (Fig. 1) shown to have frequency-dependent
dielectric permittivity

\[
\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2},
\]

strongly reminiscent of an isotropic plasma with effective plasma frequency\(^{4} \)

\[
\omega_p = \frac{(c/d)[2\pi/\ln(d/w)]}{2} \sqrt{\frac{1}{\varepsilon_r}}
\]
determined by wire thickness \( w \), mesh period \( d \), and the speed of light, \( c \). Because \( \varepsilon_r < 0 \) for \( \omega < \omega_p \), the existence of surface waves at the interface between the
3D wire mesh and vacuum can be expected.\(^{4} \) Short-

wavelength plasma surface waves at \( \omega = \omega_p/\sqrt{2} \) (and \( \varepsilon_r = -1 \)) are important in setting the resolution limit
of a superlens.\(^{5} \)

It is demonstrated in this Letter that the description of the wire mesh metamaterial as a plasma is
strongly modified by spatial dispersion (SD). Strong SD of a two-dimensional (2D) array of metallic wires
has been predicted\(^{6,9,10} \) but has not been correctly quantified for the far more interesting 3D wire mesh.
The surface waves are especially strongly influenced by SD. Quantification of the SD, numerical computa-
tion of the modified surface waves, and analytical derivation of the SD influence on surface waves con-
stitute the main results of this Letter. Surface waves at the wire mesh–vacuum interface do indeed exist,
but their dispersion relation \( \omega \) versus \( k \) is strongly

modified: the surface mode frequencies extend from
\( \omega = 0 \) to \( \omega = \omega_p \) instead of to \( \omega = \omega_p/\sqrt{2} \) as in isotropic cold plasma.

Fully 3D electromagnetic simulations are carried out using HFSS.\(^{11} \) A cubic unit cell of the 3D wire
mesh, shown in Fig. 2, consists of three intersecting square wires of width \( w \) and length \( d \), also equal to
the lattice period. To calculate the frequency of a bulk mode propagating with the wavenumber \( k = k_x e_x + k_y e_y + k_z e_z \), the boundary conditions phase shifted by \( \phi_i = k_i d \) (\( i = x,y,z \)) are set on three pairs of cube faces. All three phase advances \( \phi_i \) are varied to cal-
culate the complete Brillouin diagram. According to the common notation for a simple cubic lattice, the
center and the boundaries of the reduced Brillouin zone are labeled by four special points in the wave-
number \( k \) space: \( \Gamma(0,0,0), X(\pi/d,0,0), M(\pi/d,\pi/d,0), \) and \( R(\pi/d,\pi/d,\pi/d) \). The reduced
Brillouin zone is traversed as follows:

\( \Gamma \rightarrow X \rightarrow M \rightarrow \Gamma \rightarrow R \rightarrow X \).

Fig. 1. (Color online) Metamaterial consisting of a mesh of crossing metallic wires.
Fig. 2. (Color online) Unit cell of the wire mesh. Square wire cross section of width \( w = 0.02 \) cm, period \( d = 0.58 \) cm. Left, plasmon (longitudinal) mode. \( \vec{E} \)-vectors are plotted in the symmetry planes. Amplitude ranges from 1 (red) to 0.1 (blue). Right, photon (transverse) mode.

Fig. 3. (Color online) Brillouin diagram for the wire mesh metamaterial. Mesh parameters: same as in Fig. 2. The free space dispersion lines a are shown as blue dashed lines. The green line b is the dispersion curve of the surface wave at the metamaterial–vacuum interface. Dashed red lines c are dispersion curves of the medium with SD.

We choose the following mesh parameters: lattice period \( d = 0.58 \) cm and square wire width \( w = 0.02 \) cm \((w/d = 0.034)\). The Brillouin diagram in Fig. 3 shows the frequencies of the modes calculated as a function of the \( \vec{k} \)-vector. The effective plasma frequency, \( f_p = \omega_p/(2 \pi) = 12.97 \) GHz, is equal to the frequency of a degenerate triplet of plasmon and photon modes at the \( \Gamma \) point of the Brillouin zone. The plasma wavelength \( c/f_p \) is four times greater than the period \( d \).

Figure 2 (left) shows the electric field vectors and intensity map in the symmetry planes. The mode in Fig. 2 is calculated at the frequency of 13.0 GHz for the phase advance of \((15°, 0, 0)\). The mode is classified as a plasmon: a longitudinal wave with the electric field predominantly in the \( x \) direction of propagation. There are two degenerate photon modes that are polarized transversely to \( x \). The photon mode at the frequency of 13.15 GHz calculated for the phase advance \((15°, 0, 0)\) is plotted in Fig. 2 (right). In other words, in the region \( \Gamma – M \) the phase advances in the \( x \) and \( y \) directions are equal, and there is a longitudinal plasmon mode and two nondegenerate photon modes (ordinary \( o \) and extraordinary \( e \)), as shown in Fig. 3. The two photon modes are degenerate in the \( \Gamma – R \) direction.

Analytically, the wire mesh metamaterial is modeled as a cubic crystal with SD. The tensor of the dielectric permittivity has the following elements:\(^{12}\):

\[
\varepsilon_{xx} = \varepsilon_p + \alpha_1(k_z^2c^2/\omega^2) + \alpha_2(k_z^2c^2/\omega^2),
\]

\[
\varepsilon_{zz} = \varepsilon_p + \alpha_2(k_z^2c^2/\omega^2) + \alpha_1(k_z^2c^2/\omega^2),
\]

\[
\varepsilon_{xy} = \varepsilon_{yx} = 2\alpha_3(k_zk_zc^2/\omega^2),
\]

where the \( \vec{k} \)-vector is assumed to be in the \( x–z \) plane \((k_z = 0)\). For \( \vec{k} = 0 \) the medium is isotropic with a plasma dielectric constant \( \varepsilon_p(\omega) \). For finite \( \vec{k} \) the properties of the propagating waves are determined by the plasma frequency and the frequency-dependent SD coefficients \( \alpha_i(\omega) \), \( i = 1, 2, 3 \).

The coefficients \( \alpha_i(\omega) \) are determined by fitting the numerically calculated frequencies to the following expression\(^{12}\) that is valid near the \( \Gamma \) point:

\[
\omega^2 = \omega_p^2 + Ak^2c^2,
\]

where \( A \) depends on the propagation direction. The values of \( A \) as functions of \( \alpha_i \) are listed in Table 1. The fitting parabolic dispersion curves given by Eq. (4) are plotted in Fig. 3 as dashed red lines. They correspond to \( \alpha_1 = -0.3, \alpha_2 = 0.055, \) and \( \alpha_3 = -0.23 \).

The parameters \( \alpha_i(\omega) \) have been calculated for \( 0.017 < w/d < 0.085 \) and plotted in Fig. 4. The wire mesh metamaterial remains anisotropic for a ratio \( w/d \) as small as 0.017. A fully symmetric left-handed metamaterial\(^{13}\) using a cubic wire mesh and split rings might be more isotropic because its unit cell is more subwavelength.

The effect of the SD on wave propagation is strongly dependent on the nature of the wave. For example, the propagation of an \( x \)-polarized photon in the \( z \) direction is affected only by \( \alpha_3 \) (Table 1), which is small for \( w \ll d \). That is why recent simulations\(^{14}\) aimed at extracting \( \varepsilon_{xx}(\omega) \) of the permittivity tensor lead to the conclusion that \( \varepsilon_{xx}(\omega) \approx \varepsilon_p(\omega) \) is an accurate approximation. Surface waves, as shown below, are affected by \( \alpha_1 \) and \( \alpha_3 \), which remain significant even for \( w \ll d \). If the wire mesh metamaterial’s permittivity tensor were approximately given by \( \varepsilon_p(\omega) = \delta_{ij}\varepsilon_p(\omega) \), then one would expect an electrostatic surface wave resonance at \( f = f_p^{1/2} = 9.2 \) GHz for the parameters of Fig. 3. Our simulations indicate that is not the case.

The surface mode on a vacuum–metamaterial interface, shown in the \( x–z \) plane in Fig. 5, is calculated for different phase advances \( \phi_x \) and \( \phi_y \). The structure consisting of one cell in the \( x \) direction and five vertically stacked cells accurately models a semi-infinite space in \( z \geq 0 \) wire mesh. The surface wave \( \vec{k} \)-vector is in the \( \Gamma – X \) region. Assuming that period \( d \) is infinitesimal, the electrostatic surface wave corresponds to the \( X \) point. At the \( X \) point the frequency is significantly higher than \( f_p/\sqrt{2} \). The dispersion

<table>
<thead>
<tr>
<th>Region</th>
<th>Plasmon</th>
<th>Photon</th>
</tr>
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<tbody>
<tr>
<td>( \Gamma – X )</td>
<td>(-\alpha_1)</td>
<td>(1 – \alpha_2)</td>
</tr>
<tr>
<td>( \Gamma – M )</td>
<td>(-\alpha_1/2 + \alpha_2/2 + \alpha_3 (x))</td>
<td>(1 – \alpha_1/2 – \alpha_2/2 + \alpha_3 (x))</td>
</tr>
<tr>
<td>( \Gamma – R )</td>
<td>(-\alpha_1 + 4\alpha_2 + 6\alpha_3/3)</td>
<td>(1 – \alpha_1/3 – 2\alpha_2/3 + 2\alpha_3/3)</td>
</tr>
</tbody>
</table>
Following boundary conditions for the ing the continuity of the normal Poynting flux, the on the interface can be formulated:

In an isotropic plasma \( \varepsilon_p = -1 \) both field components \( E_x \) and \( E_z \) are equal and decay as \( \exp(-k_p|x|) \) into the plasma and vacuum. In the wire mesh metamaterial, \( E_z \) and the decay constant into the metamaterial reach their maxima at \( \varepsilon_p = -1 \) and decrease to 0 at \( \varepsilon_p = 0 \). This results in the convergence of the surface and bulk modes for large phase shifts per cell as observed in the simulations and can be seen in Fig. 3. This has important implications for superlensing by limiting the resolution: faithful reconstruction of the evanescent modes requires that the decay constants into the vacuum and into the negative permittivity material be equal to each other at the electrostatic resonance.

In conclusion, we have quantified the spatial dispersion in a wire mesh metamaterial. The SD has the most profound effect on surface waves at the metamaterial–vacuum interface. Numerical and analytical results indicate that the surface wave resonances and the spatial evanescence constant are strongly modified by the SD, with profound implications for superlensing applications of metamaterials.

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