Abstract. The most common method for treating wave propagation in tokamaks in the lower hybrid range of frequencies (LHRF) has been toroidal ray tracing, owing to the short wavelengths (relative to the system size) found in this regime. Although this technique provides an accurate description of 2D and 3D plasma inhomogeneity effects on wave propagation, the approach neglects important effects related to focusing, diffraction, and finite extent of the RF launcher. Also, the method breaks down at plasma cutoffs and caustics. Recent adaptation of full-wave electromagnetic field solvers to massively parallel computers [1] has made it possible to accurately resolve wave phenomena in the LHRF. One such solver, the TORIC code, has been modified to simulate LH waves by implementing boundary conditions appropriate for coupling the fast electromagnetic and the slow electrostatic waves in the LHRF. In this frequency regime the plasma conductivity operator can be formulated in the limits of unmagnetized ions and strongly magnetized electrons, resulting in a relatively simple and explicit form. Simulations have been done for parameters typical of the planned LHRF experiments on Alcator C-Mod, demonstrating fully resolved fast and slow LH wave fields using a Maxwellian non-relativistic plasma dielectric. Significant spectral broadening of the injected wave spectrum and focusing of the wave fields have been found, especially at caustic surfaces. Comparisons with toroidal ray tracing have also been done and differences between the approaches have been found, especially for cases where wave caustics form. The possible role of this diffraction-induced spectral broadening [2] in filling the spectral gap in LH heating and current drive will be discussed.

1. INTRODUCTION

Electromagnetic waves in the the lower hybrid range of frequencies (LHRF) are characterized by a frequency approximately at the geometric mean of the ion and electron fundamental cyclotron frequencies, $\Omega_{ci} \ll \omega \ll \Omega_{ce}$. At these frequencies, the wavelengths are on the order of a few millimeters at values of the dielectric constant, $(\omega_{pe}/\Omega_{ce})^2$, typical parameters of fusion research devices such as Alcator C-Mod and the proposed International Tokamak Experimental Reactor (ITER), where they have been proposed as a method of edge current profile control [3]. Three waves are supported by the plasma
at this frequency range: the slow lower hybrid wave (usually the one referred to as the lower hybrid wave), the fast lower hybrid wave (or Whistler wave), and the ion plasma wave which is supported by finite ion Larmor radius effects. As explained in Section 2, at plasma parameters of experimental interest that are used in the simulations presented in this paper, the ion plasma wave is strongly evanescent, and so only two waves are supported.

The short wavelength relative to machine sizes of the order of a meter have necessitated the use of ray tracing as the primary method calculating the power and current drive deposition in the plasma. While this can explain some features such as broadening of the launched spectrum due to toroidicity [4], it breaks down in cases where the rays undergo multiple reflections from cutoffs and caustics and form a stochastic field. Extended ray tracing techniques such as the Maslov method popular in seismology [5] and the wave-kinetic method [6], are valid at the caustic surfaces; but because the LH cutoffs in tokamak plasmas occur in the plasma edge where the gradients are very large, they violate the Wentzel, Kramers and Brillouin (WKB) approximation where the plasma is changing on the same scale as the wavelength [7]. There is the additional difficulty of treating linear mode conversion between the fast and slow branches, but interesting work on this is being pursued by Tracy, Kaufman, and Jaun [8].

Full wave simulations do not have these issues and incorporate the effects of diffraction that are not included in the ray tracing model. As available computer power has increased, several efforts have been made at including full wave effects in lower hybrid simulations. Pereverzev developed a hybrid method in which the ray equations were used in the direction of the group velocity and a local wave equation was used to evolve the wave front perpendicular to it [2]. He found enhanced spectral broadening due to diffraction especially at caustic surfaces. Peysson has performed full wave lower hybrid simulations in cylindrical geometry and coupled those calculations to a two dimensional Fokker-Planck code [9]. The lack of toroidicity reduced the wave equation to one dimension with toroidal and poloidal mode numbers as parameters. Also, the simulations were applicable to large aspect ratio devices.

We present in this paper, the first full wave calculations of lower hybrid waves in toroidal geometry. The plasma dielectric is Maxwellian and non-relativistic. The electrons are treated as being strongly magnetized and the ions are unmagnetized. Thermal effects are retained. This paper is organized as follows. In Section 2 we will describe in more detail, the wave equation and dielectric model used in these simulations.

2. THE WAVE EQUATION AND PLASMA MODEL

The wave equation that results from the linearized Maxwell-Boltzmann system [10, 11] is given in Eq. (1). The plasma response is embodied in the term \( J^P \) in Eq. (2). We have formulated the plasma conductivity, \( \sigma \), for the LHRF in which the lower hybrid frequency is given as, \( \omega_{LH} = \omega_{pi}/\sqrt{1 + (\omega_{pe}/\Omega_{ce})^2} \), for which the relation, \( \Omega_{ci} \ll \omega \sim \omega_{LH} \ll \Omega_{ce} \) holds.
FIGURE 1. Cold plasma electromagnetic dispersion relation for slow and fast LH waves using Alcator C-Mod parameters: [deuterium gas, \( n_i = 1.5 \), \( f_0 = 4.6 \text{ GHz} \), \( B_0 = 5.3 \text{ T} \), \( T_e = 3.5 \text{ keV} \), \( T_i = 2.0 \text{ keV} \), \( I = 1 \text{ MA} \), \( n_e(0) = 1.5 \times 10^{20} \text{ m}^{-3} \).] The accessibility plot in the inset shows that the waves are excluded from the region between 3 and 10 cm. In the full size graph, the corresponding dispersion relation for this case shows that the two propagating branches are heavily damped in this excluded region. The lines without symbols are the fast branch and the lines with symbols are the slow branch. Solid lines are the real part of \( n_\perp^2 \) and the corresponding dashed lines are the imaginary part with the sign reversed.

\[
\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \left\{ \mathbf{E} + \frac{4\pi i}{\omega} \left( J^p + J^A \right) \right\}
\]

\[
J^p = \sigma [f_0(x, v_\perp, v_\parallel)] \cdot \mathbf{E}
\]

In this range of frequencies, the ions are unmagnetized and the electrons are strongly magnetized \([k_\perp \rho_e^2 \ll 1]\). Thus, we no longer need the ion finite Larmor radius (FLR) effects and the LH plasma dielectric, \( \varepsilon \equiv 1 + \frac{4\pi i}{\omega} \sigma \), simplifies to the zero FLR contributions:

\[
\sigma \cdot \mathbf{E} = S \mathbf{E}_\perp + iD (\mathbf{b} \times \mathbf{E}_\perp) + P E_z \mathbf{b}
\]

\[
S \approx 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2} \approx 1
\]

\[
D \approx -\frac{\omega_{pe}^2}{\omega^2} \frac{\omega}{\Omega_{ce}} + \frac{\omega_{pi}^2 \Omega_{ci}}{\omega^2} \approx -\frac{\omega_{pe}^2}{\omega \Omega_{ce}}
\]

\[
P = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \approx \frac{\omega_{pe}^2}{\omega^2}
\]

where \( S \), \( D \), and \( P \), are the Stix cold plasma dielectric elements in the LHRF for the normal, co-normal, and parallel directions [12]. The neglected FLR corrections to Eq. (3)
support the mode converted ion plasma wave. Since this wave does not propagate for plasmas of experimental interest in which \( \omega/\omega_{\text{LH}} > 2 \) [13], we may neglect these corrections in the following analysis, and thus there are only two propagating modes, the fast electromagnetic LH branch that damps via electron Landau damping (ELD) and transit time magnetic pumping (TTMP) and the slow electrostatic LH branch that damps via ELD. Note, that although the FLR terms play no role, thermal effects are retained through the plasma dispersion function which is retained in the the ELD and TTMP damping in the plasma model.

The dispersion relation associated with the dielectric in Eq. (3) is given below in Eq. (5).

\[
P_4 n_\perp^4 + P_2 n_\perp^2 + P_0 = 0 \tag{5}
\]

\[
P_4 = S
\]

\[
P_2 = (S + P)(n_\parallel^2 - S) + D^2
\]

\[
P_0 = P \left[ (n_\parallel^2 - S)^2 - D^2 \right]
\]

The spectral representation employed in \textsc{toric} in Eq. (6) uses a truncated Fourier decomposition of the flux angles and cubic Hermite polynomial finite elements (FE) for the flux dimension, \( \psi \).

\[
E(x) = \sum_m E_m(\psi) \exp(i m \theta + i n \phi) \tag{6}
\]

We may obtain an estimate of the necessary resolution by referring again to the dispersion relation in Eq. (5). In the electrostatic limit, it reduces to Eq. (7) [13]. For the parameters given in Figure 1, we can estimate that from Eq. (4) and Eq. (7) that \( k_\perp \approx 66 \text{cm}^{-1} \) or \( \lambda_\perp \approx 1 \text{mm} \).

\[
k_\perp^2 \approx -\frac{P}{S} k_\parallel^2 \Rightarrow k_\perp \approx \frac{\omega_{pe}}{\omega} k_\parallel \tag{7}
\]

For comparison, in mode conversion cases in which the ion Bernstein wave (IBW) is present, the IBW wavelength in a typical C-Mod discharge [1] is 4 to 5 mm requiring 255 poloidal modes to resolve. This implies a need for 4 times more resolution, which results in 64 times more computational power to invert the resulting stiffness matrix.

The neglected sixth order coefficient is proportional to \( \beta \sqrt{\frac{m_e}{m_i}} \) and has no effect on the slow and fast branches. Analysis of Eq. (5) yields an accessibility criterion for the slow LH wave to propagate that is given by \( n_\parallel \geq n_{\parallel a} \) where \( n_\parallel a \) is given by Eq. (8). We will use this condition to construct a case in which a fast polarized lower hybrid wave is launched from the antenna and mode converts to a slow polarized lower hybrid wave at a surface where the radial wave number vanishes (a caustic) and is trapped between this caustic and the edge cutoff. This full wave result will be contrasted with a ray tracing simulation of the same case.

\[
n_\parallel a \equiv \frac{\omega_{pe}}{\Omega_{ce}} + \sqrt{S} \tag{8}
\]
3. LOWER HYBRID CUTOFFS AND CAUSTICS

In Figure 1, we show the cold plasma dispersion relation with \( m = 0 \) for a case in Alcator C-Mod with accessibility to the slow wave limited to a narrow region that on the low field side is between \( r \) of 5 and 10 cm. The parameters used for this simulation are similar to those that are used in Alcator C-Mod LH discharges [deuterium gas, \( n_{\parallel} = 1.5 \), \( f_0 = 4.6 \) GHz, \( B_0 = 5.3 \) T, \( T_e = 3.5 \) keV, \( T_i = 2.0 \) keV, \( I = 1 \) MA, \( n_e(0) = 1.5 \times 10^{20} \text{m}^{-3} \)]. A fast LH wave launched from the low field side will start to propagate at its cut-off at \( r = 19 \) cm (the solid line with symbols in the outer plot of Figure 1), propagate inwards to the caustic at \( r \approx 10 \) cm, mode convert to the slow LH wave (the solid line without symbols in Figure 1), propagate out to the cut-off at \( r = 23.8 \) cm, and reflect inward to the caustic again. Consequently, the waves will be trapped in an annulus as they travel poloidally. This behavior is apparent in the TORIC full wave simulation results shown in Figure 2 on the right.

We now simulate the same wave-plasma conditions with a ray tracing code, ACCOME [14]. For purposes of this comparison, we disable ACCOME’s equilibrium evolution and Fokker-Plank modules. We use the same EFIT equilibria in both codes. Figure 2 shows the result in the right figure. Rays launched from different locations in the antenna region converge to form a caustic and radially reflect towards the plasma edge where they reflect from the low density cutoff and receive an upshift in there parallel wavenumber. When the parallel index exceeds 2, the rays are detrapped and propagate towards the plasma center where they eventually damp on the hotter core electrons. The resulting power deposition is compared with the ray tracing result in Figure 3.

FIGURE 2. Left figure: Full wave TORIC results showing wave structure confined between caustic and cutoff. Same parameters as given in Figure 1. Right figure: ACCOME ray tracing simulation with same parameters as in Figure 1. Rays initially follow caustic and are trapped between caustic and cutoff. Eventually rays are detrapped by wavenumber upshifts and cutoff reflections and damp in the center of the plasma on hotter electrons.
ACCOME’s power deposition profile is located within $r/a \sim 0.6$ whereas TORIC’s is between $r/a \sim [0.65, 0.85]$. The primary cause of this difference is the diffraction that takes place at the caustic surface in the full wave code. The resulting slow down in the phase velocity bridges the spectral gap and causes all the power in the waves to be absorbed. In contrast, in the raytracing, no such large shift occurs and the rays slowly dissipate their energy in the core. An analysis of the evolution of $n_\parallel$ in the two simulation results shows significantly more upshift in the full wave case. For raytracing, the local $n_\parallel$ evolves to 2.5 on the high field side from 1.5 at the antenna due primarily to geometric effects of major radius position. The distribution of $n_\parallel$ on flux surfaces in the full wave results have a significant upshift from an averaged launched $n_\parallel$ of 2 to greater than 4 in the middle of the annulus at $r/a = 0.75$. The resulting enhanced damping at a parallel index of 4 damps all the wave energy in that region.

![LH Radial Power Deposition Profiles](image)

**FIGURE 3.** A comparison of the power deposition predicted by raytracing and full-wave codes ACCOME and TORIC. The full wave power given by the solid curve is localized completely between the caustic and the cutoff. The raytracing result shows all power deposited inside the caustic radius in the core of the plasma.

### 4. REDUCED LOWER HYBRID WAVE EQUATION

The evanescent ion plasma wave still restricts the resolution in these simulations and can cause numerical pollution [15] without sufficient resolution. This mode may be removed from the system by algebraically eliminating $E_{\psi}$ from the system. We may do this because the coefficient of $dE_{\psi}/d\psi$ is essentially zero, even when finite larmor radius effects are retained. This decouples $E_{\psi}$, leaving only two modes for which to solve. Back substitution is used to evaluate $E_{\psi}$. In addition, the boundary conditions have been generalized in Equations 9 and 10 from the previous current strap model to include arbitrary waveguide polarization. In these equations, $\Theta_w$, $\Delta_\psi$ and $\theta_0$ represent the static magnetic field pitch angle, the angular half height of the waveguide and the angular location of the waveguide, respectively. These new boundary conditions at the
plasma wall permit the code to impress $E_\parallel$ as a boundary conditions for slow LH wave launch, as well as $E_\perp$ for fast wave launch, and are used in the next section to launch slow waves in Alcator C.

\[
\frac{c}{\omega} \left[ (\nabla \times E)^{(m)}_{[\eta, \xi]} \right]_{\Psi_A} = \frac{4\pi i}{c} J^S \left[ -\sin \Theta_w, \cos \Theta_w \right] \\
\left[ E^{(m)}_{[\eta, \xi]} \right]_{\Psi_A} = -\exp(-im\theta_0) \frac{\exp(-im\Delta_0) + 1}{4m^2\Delta_0^2 - \pi^2} \left[ -\sin \Theta_w, \cos \Theta_w \right]
\]

(9) (10)

5. APPLICATION TO ALCATOR C

The new algorithm described in the previous section permits converged lower hybrid simulations at lower resolutions. The presence of the evanescent ion plasma wave required a very fine mesh for numerical stability even though the mode in not propagative. By eliminating it, we are left only with the limits imposed by the longer wavelength slow lower hybrid wave (the fast lower hybrid wave has a yet larger wavelength.) In addition, the cpu requirements for a given resolution are reduced by approximately a factor of two because the stiffness matrix has fewer unknowns.

We now apply this new algorithm to lower hybrid experiments on the now defunct Alcator C experiment [16]. The parameters of the simulation are given in Figure 4 and are correspond to a scenario in which the lower hybrid waves are accessible to the center of the device. The range of the contours in the figure have been adjusted to accentuate the filamentary structures present in the solution.

**FIGURE 4.** Full wave simulation of lower hybrid waves in Alcator C experiment. The wave parameters are chosen such that the wave is accessible to the center. Parameters of run: (240Nr x 127 Nm) H plasma, \( n_\parallel = 1.5 \), \( f = 4.6 \text{ GHz} \), \( B_0 = 10 \text{ T} \), \( T_e = 1.8 \text{ keV} \), \( T_i = 1.0 \text{ keV} \), \( n_e(0) = 0.510^{20} m^{-3} \), I=170 kA

The full wave electric field structures are also seen in toroidal ray tracing calculations for this case, where rays undergo multiple radial transits between edge cutoffs and caustics.
near the core.

6. CONCLUSIONS

The necessary algorithms for solving full wave dispersion in realistic toroidal geometries have been developed by the RF community over the past few decades. The availability first, of more powerful serial processors and later of parallel architectures has opened up a broader range of physics regimes to these codes. For the first time we can consider lower hybrid frequency regime calculations and the wide range of physics issues involved, such as the role of wave focusing and diffraction in LH spectral broadening in 2D toroidal geometries.

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REFERENCES